MAT240 – Tutorial — Nov. 30, 2006

$T: V \to W$

$\alpha = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$

α, β are bases of V, W, respectively.

$V \xrightarrow{T} W \xrightarrow{S} U$

$\gamma = \{u_1, \dots, u_k\}$ a basis of U.

$[T]_\alpha^\beta = (a_{ij})$ $\quad T(v_i) = \sum_{j=1}^{m} a_{ij} w_j$

$$[S]_\beta^\gamma [T]_\alpha^\beta = [S \circ T]_\alpha^\gamma$$

We write $[T]_\alpha = [T]_\alpha^\alpha$ if $T: V \to V$

Suppose $\alpha' = \{v_1', \dots, v_n'\}$, $\beta' = \{w_1', \dots, w_m'\}$ are bases of V and W.

What is relation between $[T]_\alpha^\beta$ and $[T]_{\alpha'}^{\beta'}$?

$$[T]_{\alpha'}^{\beta'} = [I_W \circ T \circ I_V]_{\alpha'}^{\beta'}$$

$$= \underbrace{[I_W]_\beta^{\beta'}} [T]_\alpha^\beta \underbrace{[I_V]_{\alpha'}^{\alpha}} \quad \text{By Thm 2.11}$$

change of basis matrix.

Note: they are invertible.

<u>Property</u>: $T: V_\alpha \to V_{\alpha'}$ is invertible

$$[T]_\alpha^{\alpha'} = \left([T^{-1}]_{\alpha'}^{\alpha}\right)^{-1}$$

<u>Pf</u>: $[T]_\alpha^{\alpha'} [T^{-1}]_{\alpha'}^{\alpha}$

$= [T \circ T^{-1}]_{\alpha'}^{\alpha'}$ (Thm 2.11)

$= [I_V]_{\alpha'}^{\alpha'}$

$= I$.

$\therefore$ they are inverses

Property: T as before

$$[T]_{\alpha}^{\alpha} \overset{?}{\sim} [T]_{\alpha'}^{\alpha'}$$

$$[T]_{\alpha'}^{\alpha'} = [I_V \circ T \circ I_V]_{\alpha'}^{\alpha'}$$

$$= [I_V]_{\alpha}^{\alpha'} [T]_{\alpha}^{\alpha} [I_V]_{\alpha'}^{\alpha}$$ (Thm 2.11)

$$\boxed{Q := [I_V]_{\alpha}^{\alpha'}}$$

Change of basis

$$[T]_{\alpha'} = Q [T]_{\alpha} Q^{-1}$$

$\boxed{5.1}$. 3. c) $A = \begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix} \in M_{2\times 2}(\mathbb{C})$

① Find e-value. ↑ $Av = \lambda v$ for some $v \neq 0$

Find roots of characteristic polynomial.

$$0 = \det(A - \lambda I)$$
$$= \det \begin{bmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{bmatrix}$$
$$= (i-\lambda)(-i-\lambda) - 2$$
$$= \lambda^2 + 1 - 2$$
$$= (\lambda - 1)(\lambda + 1)$$

e-values are ± 1.

② Find e-vectors

for $+1$ e-value:

$$N(A-I) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{pmatrix} i-1 & 1 \\ 2 & -i-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$+1$ eigenspace

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{pmatrix} i-1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$ by row reduction

$$= \left\{ \begin{pmatrix} x \\ (1-i)x \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; (i-1)x + y = 0 \right\}$$

$\therefore \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is a $+1$ e-vector.

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for -1 e-value:

$$N(A+I) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} i+1 & 1 \\ 2 & -i+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} i+1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ -(1+i)x \end{pmatrix} \right\}.$$

$\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ is a -1 e-vector.

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③ Diagonalization.

$$Q = \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}$$

$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} -1-i & -1 \\ -1+i & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -1-i & -1 \\ -1+i & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$$Q^{-1}AQ = -\frac{1}{2} \begin{pmatrix} -1-i & -1 \\ -1+i & 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -i+1-1 & -1 \\ -i+1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-i & -1-i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$T: V \longrightarrow V$

$\alpha = \{v_1, \dots, v_n\}$ basis of V

$\boxed{[T]_{\alpha'} = Q[T]_\alpha Q^{-1}}$ (*)

$\det T = \det[T]_\alpha$

a) Let α' be another basis and show that $\det[T]_\alpha = \det[T]_{\alpha'}$

<u>Pf.</u> by (*), $\det[T]_{\alpha'} = \det(Q[T]_\alpha Q^{-1})$

$= \det Q \det[T]_\alpha \det(Q^{-1})$

$= \cancel{\det Q} \det[T]_\alpha \cancel{\frac{1}{\det Q}}$ (b/c Q is inver

$\square$

b) easy.

c) $\det(T^{-1}) = (\det T)^{-1}$

$\det[T^{-1}]_\alpha = \det[T]_\alpha^{-1}$

d) Thm 2.11

e) $\det(T - \lambda I_V) \overset{?}{=} \det([T]_\alpha - \lambda I)$

$\det([T - \lambda I_V]_\alpha^\alpha) = \det([T]_\alpha - \lambda[I]_{\alpha'}^{\alpha})$

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$A, B \in M_{n\times n}(\mathbb{C})$ B is invertible.

Show

$A + cB$ is not invertible for some $c \in \mathbb{C}$

$$\begin{aligned} & \det(A + cB) \\ = \ & \det(AB^{-1} + cI) \cdot \det B \\ = \ & \det(-1)^n \underbrace{\det(-AB^{-1} - cI)}_{\text{characteristic polynomial of } -AB^{-1}} \cdot \det B \end{aligned}$$

∴ not constant polynomial.
over $\mathbb{C}$, it has a root

b) ~~Find A invertible~~ s.t.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A + cB = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

0. $f(t) = \det(A - tI)$.

$\Downarrow$

$\det A$.

a) $n = 2$: $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

$$\begin{aligned} f(t) = \det(A - tI) &= \det\begin{pmatrix} A_{11} - t & A_{12} \\ A_{21} & A_{22} - t \end{pmatrix} \\ &= (A_{11} - t)(A_{22} - t) \underbrace{- A_{12}A_{21}}_{q(t)} \end{aligned}$$

general n:

$$A - tI = \begin{pmatrix} A_{11}-t & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-t & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nn}-t \end{pmatrix}$$

$$= (A_{11}-t)\det\begin{pmatrix} A_{22}-t & \cdots & A_{2n} \\ \vdots & & \vdots \\ A_{2n} & \cdots & A_{nn}-t \end{pmatrix}.$$

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$\times 3$ case: $A - tI = \begin{pmatrix} A_{11}-t & A_{12} & A_{13} \\ A_{21} & A_{22}-t & A_{23} \\ A_{31} & A_{32} & A_{33}-t \end{pmatrix}$

$$\det(A-tI) = A_{11}-t \, \det\begin{pmatrix} A_{22}-t & A_{23} \\ A_{32} & A_{33}-t \end{pmatrix}$$

$$- A_{12}\det\begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33}-t \end{pmatrix} + A_{13}\det\begin{pmatrix} A_{21} & A_{22}-t \\ A_{31} & A_{32} \end{pmatrix}.$$

Fact: ~~Since cofactor is ... formed by taking one entry~~ of each row.

It's always 1 degree less.

Then show for n-case.