Lec.
31.10.06

Let $\beta = (u_1, \ldots u_n)$ be an ordered basis of V

isomorphism V to F

Defines a linear trans $\phi_\beta : V \longrightarrow F^n$ by $u_i \longmapsto e_i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ i^{th}

$\phi_\beta (u_i) = e_i$

if $v \in V$ then $v = \sum a_i u_i$

$\phi_\beta (v) = \phi_\beta (\sum a_i v_i) = \sum a_i \phi_\beta (u_i) = \sum a_i e_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Definition:

$$[v]_\beta := \phi_\beta (v) = \begin{pmatrix} a_1 \\ | \\ | \\ a_n \end{pmatrix}$$

Example:

$$[x^2-2x+3]_{(1,x,x^2)} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

in $P_2(\mathbb{R})$ $\quad x^2-2x+3 = +3 \cdot 1 - 2 \cdot x + 1 \cdot x^2$

$$[x^2-2x+3)_{(1,x^2,x)} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Let $T: V \longrightarrow W$ be a l.t.

Let $\beta = (v_1, \ldots v_n)$ be an ordered basis of V

$\gamma = (w_1 - \ldots w_n)$ ——— " ——— of W

compute Tv_1 relative to γ

find the column

Let $A = [T]_\beta^\gamma = \left([Tv_1]_\gamma \,\middle|\, [Tv_2]_\gamma \,\middle|\, \cdots \,\middle|\, [Tv_n]_\gamma \right)$

"The matrix of T relative to β and γ"

$\in M_{m\times n}(F)$

m: # of dim W (Since $Tv_1 \in W$)

n: # dim V

Examples ⓪:

$T = 0 \quad V \to W$

$$[Tv_1]_\gamma = [0]_\gamma = \left[\sum 0 w_i\right]_\gamma = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & & \end{pmatrix} = 0_{M_{m\times n}}$$

1. Take $V = W$, $\gamma = \beta$, $T = I_V \quad V \to V$

$I_V(v) = v$

$$[I_V]_\beta^\beta = \left([I_V v_1]_\beta \,\middle|\, \cdots \,\middle|\, [I_V v_n]_\beta \right)$$

$$= \left([v_1]_\beta \,\middle|\, [v_2]_\beta \,\middle|\, \cdots \,\middle|\, [v_n]_\beta \right)$$

$$= \left(e_1 \,\middle|\, e_2 \,\middle|\, e_3 \,\middle|\, \cdots \, e_n \right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix} =$$

An $n \times n$ matrix, everything is 0 except ones on diagonal

Example 2: $D : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ $D(p) = p'$

$\beta = (1, x, x^2, x^3)$ $\gamma = (1, x, x^2)$

$$[D]_\beta^\gamma = \left([D1]_\gamma \,\middle|\, [Dx]_\gamma \,\middle|\, [Dx^2]_\gamma \,\middle|\, [Dx^3]_\gamma \right)$$

$$= \left([0]_\gamma \,\middle|\, [1]_\gamma \,\middle|\, [2x]_\gamma \,\middle|\, [3x^2]_\gamma \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Example 2' $\gamma' = (x^2, x, 1)$

$[D]_\beta^\gamma$ → order of coordinates above changes since γ' (order of γ') is different

$$\Rightarrow [D]_\beta^{\gamma'} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Ex. 2'': $\beta'' = (x^3, x^2, x, 1)$

$$[D]_{\beta''}^{\gamma} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

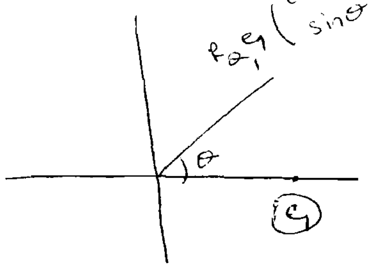
$$[D]_{\beta''}^{\gamma'} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

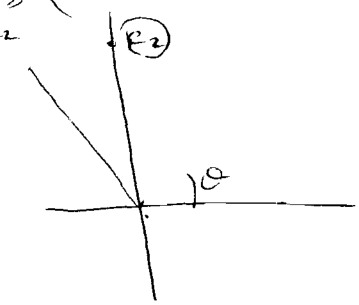
Example 3 $T = R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ "rotation by θ radians counterclockwise"

$\beta_1 = (e_1, e_2)$ $\gamma = (e_1, e_2)$

$$[R_\theta]_\beta^\gamma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$R_\theta e_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = v_1$ $R_\theta e_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = v_2$





(3') $\gamma' = (v_1, v_2)$ → choose complicated basis

 " $\quad$ "

$R_\theta e_1 \quad R_\theta e_2$

$$[R_\theta]_\beta^{\gamma'} = \Big([R_\theta e_1]_{\gamma'} \Big| [R_\theta e_2]_{\gamma'} \Big) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thm: Fix V, W, β, γ

$$T \rightsquigarrow [T]_\beta^\gamma = A$$

① T can be reconstructed from A.

② Every A arises, i.e., for every given A,

$$\exists\, T : V \to W \text{ s.t. } [T]_\beta^\gamma = A$$

Restatement: $\mathcal{L}(V,W)$ = set of all lin trans.

$$T: V \longrightarrow W$$

$$\mathcal{L}(V,W) \ni T \xmapsto[\Phi]{} [T]_\beta^\gamma \in M_{m\times n}(F)$$

$$\left.\begin{array}{l} 1.\ \Phi \text{ is 1-1.} \\ 2.\ \Phi \text{ is onto} \end{array}\right\} \Phi \text{ is invertible.}$$

Pf: $A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ then

$$A = [T]_\beta^\gamma \iff Tv_j = \sum_{i=1}^m a_{ij} w_i \quad \text{i.e} \quad Tv_1 = \sum_{i=1}^m a_{i1} w_i = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

$$Tv_2 = \sum_{i=1}^m a_{i2} w_i = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} \text{ 2nd co}$$

If you know A, you know a_{ij}'s, you know Tv_j for each j, but T is determined by its values on basis elements.

Now suppose A is given. Let T be the linear trans. for which

$$Tv_j = \sum a_{ij} w_i$$

This defines a li transformation T, and $[T]_\beta^\gamma = A$

Definition: $\mathcal{L}(V, W)$ is a v.s. with the following operations
(Theorem)

1. $O_{\mathcal{L}(V,W)}(v) = 0$

2. If $T, S : V \longrightarrow W$ are lin. transformation

$(T+S)(v) = T(v) + S(v)$

3. If $T : V \longrightarrow W$ is a lin. trans., and $c \in F$

$(cT)(v) = c \cdot T(v)$

$T + O_{\mathcal{L}(V,W)}(v) = T(v) + O(v) = T(v) + 0 = T(v) \Rightarrow T + 0 = T$ ✓

Thms $\Phi : \mathcal{L}(V, W) \longrightarrow \underbrace{M_{m \times n}}_{\text{a v.s}}(F)$ is an isomorphism of vector spaces.

Need to prove

1. Φ is 1-1 & onto

2. Φ is linear

a. $[\Phi]$ is linear

b. $[T+S]_\beta^\gamma = [T]_\beta^\gamma + [S]_\beta^\gamma$

c. $[zT]^\gamma = z \cdot [T]^\gamma$

proof of b:

$$[T]_\beta^\gamma = \left([Tv_1]_\gamma \middle| \cdots\cdots \middle| [Tv_n]_\gamma \right)$$

$$[S]_\beta^\gamma = \left([Sv_1]_\gamma \middle| \cdots\cdots \middle| [Sv_n]_\gamma \right)$$

$$[T+S]_\beta^\gamma = \left([(T+S)(v_1)]_\gamma \middle| \cdots\cdots \middle| [(T+S)(v_n)]_\gamma \right)$$

$$= \left([T(v_1)+S(v_1)]_\gamma \middle| \cdots\cdots \right)$$

$$= \left([T(v_1)]_\gamma + [S(v_1)]_\gamma \middle| \cdots\cdots \right)$$

$$= \left([T(v_1)]_\gamma + \cdots\cdots \right) + \left([S(v_1)]_\gamma + \cdots\cdots \right)$$

$$= [T]_\beta^\gamma + [S]_\beta^\gamma$$

$$\overset{u}{U} \xrightarrow[\text{lin.}]{S} \overset{S(u)}{V} \xrightarrow[\text{lin}]{T} \overset{T(S(u))}{W}$$

$\alpha = (u_1, \ldots u_k)$ $\beta = (v_1, \ldots v_n)$ $\gamma = (w_1, \ldots w_m)$

1. $[S]_\alpha^\beta = M_1$ 2. $[T]_\beta^\gamma = M_2$ 3 $[T \circ S]_\alpha^\gamma = M_3$

Claim:

$(T \circ S)(u) := T(S(u))$ is a linear transformation

$T \circ S : U \to W$

Proof of Claim:

$$(T \circ S)(7u) = T(S(7u)) \overset{S \text{ is}}{\underset{\text{lin}}{=}} T(7S(u))$$

$$\| \qquad = 7T(S(u)) = 7(T \circ S)(u) \checkmark$$

$7(T \circ S)(u)$

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$$\mathcal{L}(V, W) \longrightarrow M_{m \times n}(F)$$

dim = n m

basis = β γ

$$T \longmapsto [T]_\beta^\gamma = A$$

$$\left.\begin{array}{l} T \to A \\ S \to B \\ T+S \to A+B \\ 7T \to 7A \end{array}\right\}$$

$$A = \begin{pmatrix} a_{11} & & a_{1n} \\ [Tv_1]_\gamma & \cdots & [Tv_n]_\gamma \\ a_{\cdot\cdot} & & \vdots \end{pmatrix} \iff \forall_j \quad Tv_j = \sum_{k=1}^{m} a_{kj} w_k$$

$$\begin{tikzcd}
U \arrow[r, "S"] \arrow[rr, bend right, "T\circ S \to ?\ C"'] & V \arrow[r, "T"] & W
\end{tikzcd}$$

$(u_i)_{i=1}^{l}$ $\quad B \downarrow$ matrix B $\quad (v_j)_{j=1}^{n}$ $\quad A \downarrow$ matrix A $\quad (w_k)_{k=1}^{m}$

$$M_{m\times n} \ni A = (a_{kj}) \longrightarrow Tv_j = \sum_{k=1}^{m} a_{kj} w_k$$

$$M_{n\times l} \ni B = (b_{ji}) \longrightarrow Su_i = \sum_{j=1}^{n} b_{ji} v_j$$

$$M_{m\times l} \ni C = (c_{kj}) \longrightarrow T\circ S(u_i) = \sum_{k=1}^{m} \underbrace{c_{ki}}_{?} w_k$$

$$T\circ S(u_i) = T(S(u_i)) = T\left(\sum_{j=1}^{n} b_{ji} v_j\right) = \sum_{j=1}^{n} b_{ji}\, T(v_j)$$

$$= \sum_{j=1}^{n} b_{ji}\, T(v_j) = \sum_{j=1}^{n} b_{ji} \sum_{k=1}^{m} a_{kj} w_k$$

$$= \sum_{k=1}^{m} \underbrace{\sum_{j=1}^{n} a_{kj} b_{ji}}_{c_{ki}} w_k$$

moral: $C = (c_{ki})$

$$c_{ki} = \sum_{j=1}^{n} a_{kj} b_{ji}$$