

Problem Set 19 — MAT257

April 5, 2017

Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

1 Munkres §37 (pp.308–309)

4. Let M be the 2-manifold in \mathbb{R}^3 consisting of all points \mathbf{x} such that

$$4(x_1)^2 + (x_2)^2 + 4(x_3)^2 = 4 \quad \text{and} \quad x_2 \geq 0.$$

Then ∂M is the circle consisting of all points such that

$$(x_1)^2 + (x_3)^2 = 1 \quad \text{and} \quad x_2 = 0.$$

See Figure 37.5 (Munkres, p.309). The map

$$\alpha(u, v) = (u, 2\sqrt{1 - u^2 - v^2}, v),$$

for $u^2 + v^2 < 1$, is a coordinate patch on M that covers $M \setminus \partial M$. Orient M so that α belongs to the orientation, and give ∂M the induced orientation.

- (a) What normal vector corresponds to the orientation of M ? What tangent vector corresponds to the induced orientation of ∂M ?
- (b) Let ω be the 1-form $\omega = x_2 dx_1 + 3x_1 dx_3$. Evaluate $\int_{\partial M} \omega$ directly.
- (c) Evaluate $\int_M d\omega$ directly, by expressing it as an integral over the unit disc in the (u, v) plane.

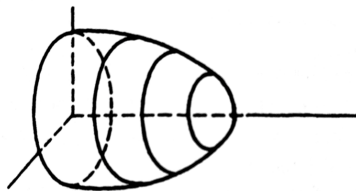


Figure 37.5

5. The 3-ball $B^3(r)$ is a 3-manifold in \mathbb{R}^3 ; orient it naturally and give $S^2(r)$ the induced orientation. Assume that ω is a 2-form defined in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that

$$\int_{S^2(r)} \omega = a + \frac{b}{r}$$

for each $r > 0$.

- (a) Given $0 < c < d$, let M be the 3-manifold in \mathbb{R}^3 consisting of all \mathbf{x} with $c \leq \|\mathbf{x}\| \leq d$, oriented naturally. Evaluate $\int_M d\omega$.
 - (b) If $d\omega = 0$, what can you say about a and b ?
 - (c) If $\omega = d\eta$ for some η in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, what can you say about a and b ?
6. Let M be an oriented $(k+l+1)$ -manifold without boundary in \mathbb{R}^n . Let ω be a k -form and let η be an l -form, both defined in an open set of \mathbb{R}^n about M . Show that

$$\int_M \omega \wedge d\eta = a \int_M d\omega \wedge \eta$$

for some a , and determine a . (Assume M is compact.)

2 Munkres §38 (pp.320–322)

2. Let G be a vector field defined in $A = \mathbb{R}^n \setminus \{\mathbf{0}\}$ with $\text{div } G = 0$ in A .
- (a) Let M_1 and M_2 be compact n -manifolds in \mathbb{R}^n , such that the origin is contained in both $M_1 \setminus \partial M_1$ and $M_2 \setminus \partial M_2$. Let N_i be the unit outward normal vector field to ∂M_i , for $i = 1, 2$. Show that

$$\int_{\partial M_1} \langle G, N_1 \rangle dV = \int_{\partial M_2} \langle G, N_2 \rangle dV.$$

[*Hint:* Consider for the case where $M_2 = B^n(\epsilon)$ and is contained in $M_1 \setminus \partial M_1$. See Figure 38.2 (Munkres, p.321).]

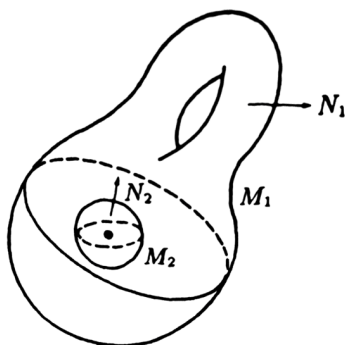


Figure 38.2

- (b) Show that as M ranges over all compact n -manifolds in \mathbb{R}^n for which the origin is not in ∂M , the integral

$$\int_{\partial M} \langle G, N \rangle dV,$$
 where N is the unit normal to ∂M pointing outwards from M , has only two possible values.
3. Let G be a vector field in $B = \mathbb{R}^n \setminus \{\mathbf{p}, \mathbf{q}\}$ with $\text{div } G = 0$ in B . As M ranges over all compact n -manifolds in \mathbb{R}^n for which $\{\mathbf{p}\}$ and $\{\mathbf{q}\}$ are not in ∂M , how many possible values does the integral

$$\int_{\partial M} \langle G, N \rangle dV$$

have? (Here N is the unit normal to ∂M pointing outwards from M .)

5. Let S be the subset of \mathbb{R}^3 consisting of the union of:

- i the z -axis,
- ii the unit circle $x^2 + y^2 = 1, z = 0$,
- iii the points $(0, y, 0)$ with $y \geq 1$.

Let A be the open set $\mathbb{R}^3 \setminus S$ of \mathbb{R}^3 . Let C_1, C_2, D_1, D_2, D_3 be the oriented 1-manifolds in A that are pictured in Figure 38.3 (Munkres, p.322). Suppose that F is a vector field in A , with $\text{curl } F = \mathbf{0}$ in A , and that

$$\int_{C_1} \langle F, T \rangle ds = 3 \quad \text{and} \quad \int_{C_2} \langle F, T \rangle ds = 7.$$

What can you say about the integral

$$\int_{D_i} \langle F, T \rangle ds$$

for $i = 1, 2, 3$? Justify your answers.

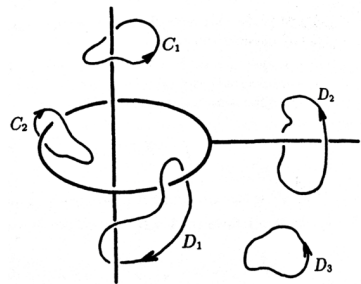


Figure 38.3

3 “Ponder...”

Challenge! Why does a planimeter work? See images at <http://drorbn.net/AcademicPensieve/Classes/1617-257b-AnalysisII/Planimeter/>

