Solution to Homework 3

15.12 To show they are isomorphic, construct an isomorphism:

$$\Phi: \mathbb{Z}[\sqrt{2}] \rightarrow H \text{ by } \phi(a+b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$$

To show that $$\Phi$$ is a homomorphism is trivial (check $$\phi(ab) = \phi(a)\phi(b)$$, $$\phi(a+b) = \phi(a) + \phi(b)$$).

To show $$\Phi$$ is an isomorphism, show that $$\Phi$$ is one-to-one and onto.

i.e. if $$\phi(x) = \phi(y)$$ then

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$$

And $$x = y$$

Therefore $$\Phi$$ is one-to-one.

Let $$h \in H$$ (arbitrary). Then $$h = \begin{bmatrix} m & 2n \\ n & m \end{bmatrix}$$

Then show that there exists some $$k \in \mathbb{Z}[\sqrt{2}]$$ such that $$\phi(k) = h$$. In particular, $$k = m+n\sqrt{2}$$.

Therefore, for all $$h \in H$$, there exists some $$k \in \mathbb{Z}[\sqrt{2}]$$ such that $$\phi(k) = h$$ and therefore $$F$$ is onto.

Therefore by the definition of an isomorphism, $$\Phi$$ is an isomorphism

15.15 Let $$\Phi: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$$ by $$\phi(x) = 6x$$.

$$\mathbb{Z}_5$$ is a partition of $$\mathbb{Z}$$ based on equivalency classes modulo 5. Therefore for all $$x \in \mathbb{Z}$$,

$$[x] = \{x + 5k \mid k \in \mathbb{Z}\}$$.

Therefore the elements of $$\mathbb{Z}_5$$ can be interpreted as equivalence classes of $$\mathbb{Z}$$. Similarly for $$\mathbb{Z}_{30}$$.

Notation: $$[x]_y$$ will denote the equivalence class of $$x$$ modulo $$y$$

Therefore: $$\Phi: \{\text{equivalency classes modulo } 5\} \rightarrow \{\text{equivalency classes modulo } 30\}$$ by $$\phi([x]_5) = [6x]_{30}$$

To show that $$\Phi$$ is a homomorphism (let $$z \in \mathbb{Z}$$)

Let $$x, y \in \mathbb{Z}_5$$:

$$\phi([x]_5 + [y]_5) = \phi([x+y+5z]) = [6x+6y+30z]_{30} = [6x]_{30} + [6y]_{30} = \phi([x]_5) + \phi([y]_5)$$

$$\phi([x]_5)[y]_5) = \phi([xy+5z]) = [6xy+30z]_{30} = [6xy]_{30} = [6x]_{30} [6y]_{30} = \phi([x]_5) \phi([y]_5)$$

And therefore $$\Phi$$ is a ring homomorphism.

15.20 Homomorphisms from $$\mathbb{Z}_6 \rightarrow \mathbb{Z}_6$$:

As in Q15, $$\mathbb{Z}_6$$ is a partition of $$\mathbb{Z}$$ modulo 6. Therefore the elements of $$\mathbb{Z}_6$$ are equivalent to their equivalency classes. Furthermore, note that a homomorphism from $$\mathbb{Z}_n \rightarrow \mathbb{Z}_m$$ is fully defined by the image of 1 because all elements of $$\mathbb{Z}_n$$ are obtainable from 1. Therefore, a homomorphism from $$\mathbb{Z}_n \rightarrow \mathbb{Z}_m$$ must be of the form $$f(x) \rightarrow ax$$ for some $$a$$ in $$\mathbb{Z}$$ (i.e. $$f(1) = a \rightarrow f(1+1) = f(2) = 2f(1) = 2a$$ etc.)

Therefore, let $$F: \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$$ be a homomorphism by $$f(x) = ax$$ for some $$a$$ in $$\mathbb{Z}$$.

Then for all $$x, y$$ in $$\mathbb{Z}_6$$:

$$f(x+y) = f(x) + f(y)$$

$$f([x]+[y]) = f([x+y+6z]) = [ax+ay+6az] = [ax+ay] = [ax] + [ay] = f(x) + f(y)$$

Therefore, by the nature of the spaces we have chosen ($$\mathbb{Z}_6$$), $$f(x+y) = f(x) + f(y)$$ for any choice of $$f(1)$$.

Also,

$$f(xy) = f(x)f(y)$$

$$f([xy]) = f([xy+6z]) = [axy+6az] = [axy]$$

And, $$f([x])f([y]) = [ax][ay] = [a^2xy]$$

Therefore, $$[axy] = [a^2xy]$$ if $$F$$ is a homomorphism

$$a^2 = a \mod 6$$ (or equivalently $$[a^2] = [a]$$)

A simple check reveals that only $$a = 0, 1, 3, 4$$ satisfy this requirement.

Therefore $$F: \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$$ a homomorphism means that $$f(x) = ax$$ for some $$a$$ in $$\{0, 1, 3, 4\}$$.
Homomorphisms from $\mathbb{Z}_{20} \rightarrow \mathbb{Z}_{30}$

From the previous, a homomorphism must be of the form $f(x) = ax$; and also, we can think of elements as equivalence classes. Notation will be borrowed from previous classes.

Let $G: \mathbb{Z}_{20} \rightarrow \mathbb{Z}_{30}$ be a homomorphism by $g([x]_{20}) = [ax]_{30}$

To be a homomorphism:

$g([x+y]_{20}) = g([x]_{20}) + g([y]_{20})$

Therefore: $[ax+ay+20ak]_{30} = [ax+ay+20a(l+m)]_{30}$ for all integers $k,l,m$.

Therefore $[20a]_{30} = [0]_{30}$ (the only way $20ar = 20as$ mod 30 for all integers $r,s$ is if $20a = 0$)

Therefore, $a = 3w, w$ in $\mathbb{Z}$.

Also,

$g([xy]_{20}) = g([x]_{20})g([y]_{20})$

Therefore: $[axy]_{30} = [a^2xy]_{30}$

Therefore, $[a^2]_{30} = [a]_{30}$

In particular, if $a^2 = a$ mod 30, then $a^2 = a$ mod 10

Therefore, $a \in \{0,1,5,6,10,11,15,16,20,21,25,26\}$ (the only units digits that stay the same on squaring). Also from above, $3 | a$ and so:

$a \in \{0,6,15,21\}$ (Note that $-1 < a < 30$ since we can reduce modulo 30)

Therefore there are four homomorphisms, all given by $f(x) = ax$ for the 4 $a$'s above.

15.50

Let $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} | a, b \in \mathbb{Q}\}$, $\mathbb{Q}[\sqrt{5}] = \{a+b\sqrt{5} | a, b \in \mathbb{Q}\}$

If the rings are isomorphic then there would exist some isomorphism, $F$, between them.

The unity of both rings is the integer 1. Since $F$ is an isomorphism by assumption, it is onto and a homomorphism and therefore by Theorem 15.1.6 $f(1) = 1$. Assume further that $f(\sqrt{2}) = a$ for some $a \in \mathbb{Q}[\sqrt{5}]$.

Then, since $F$ is an isomorphism (construction of a contradiction; there are many):

$f(1+\sqrt{2}) = f(1) + f(\sqrt{2}) = 1 + a$

$f((1+\sqrt{2})(1+\sqrt{2})) = f(1+2\sqrt{2}) = f(3+2\sqrt{2}) = f(3) + f(2) = 3f(1) + 2f(1) = 3+2a$

Also, $f((1+\sqrt{2})^2) = f((1+\sqrt{2})^2) = (1+a)^2 = 1+2a+a^2$

Therefore: $a^2 + 2a + 1 = 3 + 2a$

$a^2 - 2a = 0$

$(a+\sqrt{2})(a-\sqrt{2}) = 0$

$a = \pm \sqrt{2}$

But $\sqrt{2}$ is not in $\mathbb{Q}[\sqrt{5}]$. A contradiction.

Therefore no such $F$ exists.

Therefore the two spaces are not isomorphic.