Solution to Homework 3

15.12 To show they are isomorphic, construct an isomorphism:

Φ:Z[$\sqrt{2}$]$\rightarrow$H by $\phi(a+b\sqrt{2})=$ $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$

To show that Φ is a homomorphism is trivial (check $\phi(ab)= \phi(a)\phi(b)$, $\phi(a+b)=\phi(a)= \phi(b)$)

To show Φ is an isomorphism show that Φ is one-to-one and onto.

i.e. if $\phi(x)= \phi(y)$ then $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$

And x=y

Therefore Φ is one-to-one.

Let h e H (arbitrary). Then h= $\begin{bmatrix} m & 2n \\ n & m \end{bmatrix}$

Then show that there exists some k e Z[$\sqrt{2}$] such that $\phi(k)=h$. In particular, $k=m+n\sqrt{2}$.

Therefore, for all h e H, there exists some k e Z[$\sqrt{2}$] such that $\phi(k)=h$ and therefore F is onto.

Therefore by the definition of an isomorphism, Φ is an isomorphism

15.15 Let $\Phi:Z_5\rightarrow Z_5$ by $\phi(x)=6x$.

Z_5 is a partition of Z based on equivalency classes modulo 5. Therefore for all x e Z, $[x]=\{x+5k \mid k e Z\}$.

Therefore the elements of Z_5 can be interpreted as equivalence classes of Z. Similarly for Z_{30}.

Notation: $[x]_y$ will denote the equivalence class of x modulo y

Therefore: Φ: {equivalency classes modulo 5}$\rightarrow${equivalency classes modulo 30} by $\phi([x]_5)=[6x]_{30}$

To show that Φ is a homomorphism (let z e Z)

Let x, y e Z_5:

$\phi([x]_5+[y]_5)=\phi([x+y]_5)=\phi([x+y+5z])=[6x+6y+30z]_{30}=[6x+6y]_{30}=[6x]_{30}+[6y]_{30}= \phi([x]_5)+ \phi([y]_5)$

$\phi([x]_5[y]_5)=\phi([xy+5z]_5)=[6xy+30z]_{30}=[6xy]_{30}=[36xy]_{30}=[6x]_{30}[6y]_{30}= \phi([x]_5)\ \phi([y]_5)$

And therefore Φ is a ring homomorphism.

15.20 Homomorphisms from $Z_6\rightarrow Z6$:

As in Q15, Z_6 is a partition of Z modulo 6. Therefore the elements of Z_6 are equivalent to their equivalency classes. Furthermore, note that a homomorphism from $Z_6\rightarrow Z_6$ is fully defined by the image of 1 because all elements of Z_6 are obtainable from 1. Therefore, a homomorphism from $Z_m\rightarrow Z_n$ must be of the form $f(x)\rightarrow ax$ for some a in Z (i.e. $f(1)=a \rightarrow f(1+1)=f(2)=2f(1)=2a$ etc.)

Therefore, let $F:Z_6\rightarrow Z_6$ be a homomorphism by $f(x)=ax$ for some a in Z.

Then for all x,y in Z_6:

$f(x+y)=f(x)+f(y)$

$f([x]+[y])=f([x+y+6z])=[ax+ay+6az]=[ax+ay]=[ax]+[ay]=f(x)+f(y)$

Therefore, by he nature of the spaces we have chosen (Z_6), $f(x+y)=f(x)=f(y)$ for any choice of f(1)

Also,

$f(xy)=f(x)f(y)$

$f([xy])=f([xy+6z])=[axy+6az]=[axy]$

and, $f([x])f([y])=[ax][ay]=[a^2xy]$

therefore, $[axy]=[a^2xy]$ if F is a homomorphism

$a^2 = a \bmod 6$ (or equivalently $[a^2]=[a]$)

A simple check reveals that only a=0,1,3,4 satisfy this requirement.

Therefore $F:Z_6\rightarrow Z_6$ a homomorphism means that $f(x)=ax$ for some a in {0,1,3,4}.

Homomorphisms from $Z_{20} \rightarrow Z_{30}$
From the previous, a homomorphism must be of the form $f(x)=ax$; and also, we can think of elements as equivalency classes. Notation will be borrowed from previous question.
Let $G:Z_{20} \rightarrow Z_{30}$ be a homomorphism by $g([x]_{20})=[ax]_{30}$

To be a homomorphism:
$g([x+y]_{20})=g([x]_{20})+g([y]_{20})$
Therefore: $[ax+ay+20ak]_{30} = [ax+ay+20a(l+m)]_{30}$ for all integers k,l,m.
Therefore $[20ak]_{30}=[20a(l+m)]_{30}$
Therefore $[20a]_{30}=[0]_{30}$ (the only way $20ar = 20as \bmod 30$ for all integers r,s is if $20a=0$)
Therefore, $a=3w$, w in Z.

Also,
$g([xy]_{20})=g([x]_{20})g([y]_{20})$
Therefore: $[axy]_{30}=[a^2xy]_{30}$
Therefore, $[a^2]_{30}=[a]_{30}$

In particular, if $a^2=a \bmod 30$, then $a^2=a \bmod 10$
Therefore, a e {0,1,5,6,10,11,15,16,20,21,25,26} (the only units digits that stay the same on squaring). Also from above, $3|a$ and so:
a e {0,6,15,21} (Note that $-1<a<30$ since we can reduce modulo 30)

Therefore there are four homomorphisms, all given by $f(x)=ax$ for the 4 a's above.

15.50
Let $Q[\sqrt{2}]=\{a+b\sqrt{2} \mid a,b \text{ e } Q\}$, $Q[\sqrt{5}]=\{a+b\sqrt{5} \mid a,b \text{ e } Q\}$
If the rings are isomorphic then there would exist some isomorphism, F, between them.
The unity of both rings is the integer 1. Since F is an isomorphism by assumption, it is onto and a homomorphism and therefore by Theorem 15.1.6 $f(1)=1$. Assume further that $f(\sqrt{2})=a$ for some a e $Q[\sqrt{5}]$.
Then, since F is an isomorphism (construction of a contradiction; there are many):

$f(1+\sqrt{2})=f(1)+f(\sqrt{2})=1+a$
$f((1+\sqrt{2})(1+\sqrt{2}))=f(3+2\sqrt{2})=f(3)+f(2\sqrt{2})=3f(1)+f(2)f(\sqrt{2})=3f(1)+2f(1)f(\sqrt{2})=3+2a$
Also, $f((1+\sqrt{2})^2)=(f(1+\sqrt{2}))^2=(1+a)^2=1+2a+a^2$
Therefore: $a^2+2a+1=3+2a$
$a^2-2=0$
$(a+\sqrt{2})(a-\sqrt{2})=0$
$a=\pm\sqrt{2}$
But $\sqrt{2}$ is not in $Q[\sqrt{5}]$. A contradiction.

Therefore no such F exists.
Therefore the two spaces are not isomorphic.