*** Integration, Stokes Theorem** Read Along 34, 35, 37. Hw 18 comm

M^k is oriented, α, β are positive & swap
$\omega \subseteq \operatorname{im}\alpha \cap \operatorname{im}\beta$, then

$$\int_{\mathbb{R}^k} \alpha^* \omega = \int_M^{\alpha} \omega = \int_{\mathbb{R}^k} \beta^* \omega =: \int_M \omega$$

$\int_{M \text{ oriented}} \omega$ in general. In practice, chop up M
to reasonable pieces w/ meas-0 intersections

integrable over each piece and add.

Example: Let S^2 be oriented as $\partial D^3 \subset \mathbb{R}^3$.
Let $\omega \in \Omega^2(\mathbb{R}^3)$ given by $\omega = x\, dy \wedge dz + y\, dz \wedge dx$
$+ z\, dx \wedge dy$.

Compute $\int_{S^2} \omega$.

precisely $\nu : S^2 \to \mathbb{R}^3$.
$\nu^* : \Omega^2(\mathbb{R}^3) \to \Omega^2(S^2)$
$\int_{S^2} \nu^* \omega \to \int_{S^2} \omega$

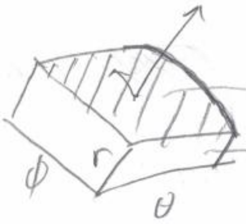
$\alpha : [0, \infty)_r \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to$
longtitude latitude

$\alpha(r,\theta,\phi) = (r\cos\theta\cos\phi,\ r\sin\theta\cos\phi)$

$\det D\alpha = r^2\cos\phi > 0$

on bulk, so α is orientation preserving, positive

$\beta = \alpha|_{r=1}$, $\beta(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\theta, \sin\phi)$



β positive.

β is positive if $[0,2\pi]\theta$, $[-\frac{\pi}{2},\frac{\pi}{2}]$ is taken (θ,ϕ).

$$\int_{S^2} w = \int_{Q=[\ ]_\theta\times[\ ]_\phi} \beta^* w$$

$$= \int_Q \underbrace{\cos\theta\cos\phi}_{x}(d(\sin\theta\cos\phi)\wedge d(\sin\phi))$$

$+2$ further big terms

$$= \int_Q \cos\phi\, d\theta\wedge d\phi = \int_{[0,2\pi]_\theta\times[-\frac{\pi}{2},\frac{\pi}{2}]} \cos\phi = 2\pi\cdot 2 = 4\pi$$

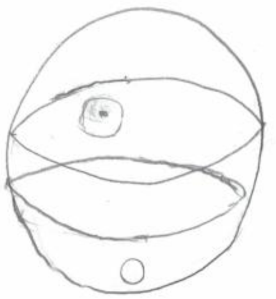
In theory

non-convergence issue...

$\int_M w$, M is compact oriented

Find a partition of unity $\phi_i : M \to [0,1]$ smooth subordinate to "positive charts of M"

* supp $\phi_i \subset \operatorname{im}(\alpha)$, α is a chart
* $\Sigma\phi_i = 1$
* local finiteness



$$\overset{\phi}{\int_M} w = \underbrace{\sum_{i\in I} \int_M \phi_i \, w}_{\text{makes sense}}$$

prop. ϕ_i, ψ_i are partition of unity,

$$\overset{\phi}{\int_M} w = \overset{\psi}{\int_M} w$$

$$\| $$

$$\int_M w$$

pf.
$$\overset{\phi}{\int_M} = \sum_{i\in I} \int_M \Big(\sum_j \psi_j\Big)\phi_i \cdot w$$
$$= \sum_i \sum_j \int_M \psi_j \cdot \phi_i \, w$$
$$= \sum_j \sum_i \int_M \phi_i \psi_j \cdot w$$
$$= \sum_j \int_M \Big(\sum \phi_i\Big)\psi_j \cdot w$$
$$= \sum_j \int_M \psi_j \cdot w$$
$$= \int_n w$$

Properties: $a_i \in \mathbb{R}$

1. $\int_M a_1 w_1 + a_2 w_2 = a_1 \int_M w_1 + a_2 \int_M w_2$

$$\int_{-M} w = -\int_M w = \int_M -w$$

M w/ opposite orientation