Feb. 6^{th} Mon Hour 050

Today: Alternating tensors. Read along: 27 – 28

Let V be a vector space with basis $\{a_1, \cdots, a_n\}$, then

Thm $\forall I \in \underline{n}^k$, $\exists !$ (unique) $\phi_I \in \mathcal{L}^k(V)$ such that $\dim \mathcal{L}^k = n^k$

$\forall J \in \underline{n}^k$, $\phi_I(a_J) = \delta_{IJ}$; $\{\phi_I\}$ is a basis of $\mathcal{L}^k(V)$

Def: $\psi \in \mathcal{L}^k(V)$ is alternating means $\psi(\cdots, x, \cdots, y, \cdots) = -\psi(\cdots, y, \cdots, x, \cdots)$ $\forall x, y$

Claim: Alternating iff $\psi(\cdots, x, \cdots x, \cdots) = 0$.

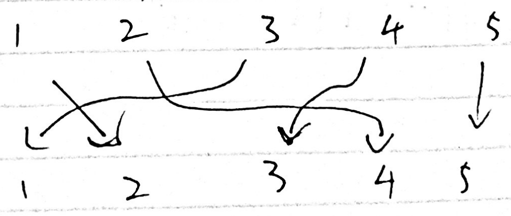
Pf: $\Rightarrow$ ✓, easy

$\Leftarrow$ $\psi(\cdots, x+y, \cdots, x+y, \cdots) = 0, = \overbrace{\psi(\cdots, x, \cdots, x, \cdots)}^{=0} + \psi(\cdots, x, \cdots, y, \cdots) + \psi(\cdots, y, \cdots, x, \cdots)$

$+ \underbrace{\psi(\cdots, y, \cdots, y, \cdots)}_{=0} = 0.$ $\square$

Def: S_n = "the permutation group of $\underline{n}$" $= \{\sigma : \underline{n} \circlearrowleft : \sigma$ 1-1, onto$\}$

Ex: ~~S_5~~ $\sigma \in S_5$.



$\sigma 1 = 2, \cdots$

$|S_n| = ! = 1\cdot 2\cdots n$, S_n is the socalled "symmetric group of order n".

$\sigma, \tau \in S_n$, $\sigma \cdot \tau = \sigma \circ \tau = (\sigma \cdot \tau)(i) = \sigma(\tau(i)) = \sigma\tau$;

Properties: 1. associative, but not commutive.

2. $\exists$ an identity.

3. Every permutation has an inverse.

Def: $A^k(V) := \{\varphi \in \mathcal{L}^k$ s.t. φ is alternating$\}$

Claim: if $\varphi \in \mathcal{L}^k(V)$, then $\varphi \in A^k(V) \iff \forall\, \sigma \in S_k$.

$$\varphi^\sigma(v_1, \cdots, v_k) := \varphi(v_{\sigma 1}, \cdots, v_{\sigma k}) = (-1)^\sigma \varphi(v_1, \cdots, v_k)$$

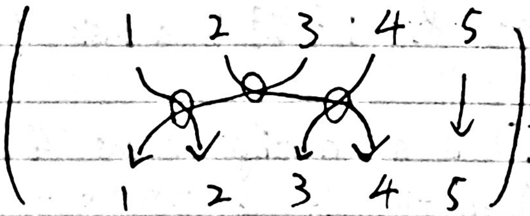
Aside: There is a unique sign assignment Sign: $S_n \to \{\pm 1\}$

usually write $\operatorname{sign}(\sigma) = (-1)^\sigma$, such that $(-1)^{ij} = (-1)$

and $(-1)^{\sigma\tau} = (-1)^\sigma \cdot (-1)^\tau$

Where $(ij) \in S_n$ with $(ij)(k) = \begin{cases} j & k = i \\ i & k = j \\ k & \text{otw.} \end{cases}$

Sketch 1.

Sign  $= (-1)^3 = -1$

: 3 Swaps

Sketch 2.

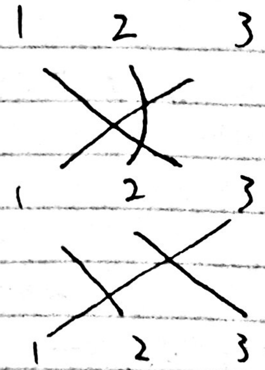
$$(-1)^\sigma := \prod_{1 \le i < j \le n} \operatorname{sign}(\sigma_j - \sigma_i).$$ NTS the property $(-1)^{ij} = -1$, ①

$(-1)^{\sigma\cdot\tau} = (-1)^\sigma \cdot (-1)^\tau$ ②

①

$(-1)^{ij}$  Sign (-1)

②



$\sigma \in S_n \sim A_\sigma \in M_{n\times n}$ Ex $n=2$ $A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ~

$$A_\sigma = \left(e_{\sigma_1} \middle| e_{\sigma_2} \middle| \cdots \middle| e_{\sigma_n} \right)$$

$(-1)^\sigma = \det(A_\sigma) \cdots$ pf: obvious. (swap till get desired results)

$\mathcal{L}^k \to A^k$ $\quad \underline{n}^k \rightsquigarrow \underline{n}^k_a = \{(i_1, \dots, i_k) \in \underline{n}^k : i_1 < i_2 < \cdots < i_k\} = \binom{n}{k}$

ψ ϕ (alternating ascending) (if $i_j = i_{j+1}$, it 0)

$1, 2, 3, 4, \cdots n$ $\quad |\underline{n}^k_a| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$i_1 \quad i_2 \quad \cdots \quad i_k$ $\quad$ number $\quad = |\binom{n}{k}|$

Thm: $\forall I \in \binom{n}{k}\ \exists!\ \psi_I \in A^k(V)$ such that $\forall J \in \binom{n}{k}$, $\psi_I(A_J) = \delta_{IJ}$;
furthermore, $\{\psi_I\}_{I \in \binom{n}{k}}$ is a basis of $A^k(V)$, $\dim A^k(V) = \binom{n}{k}$