Note: this notation: $\subset = \subseteq$

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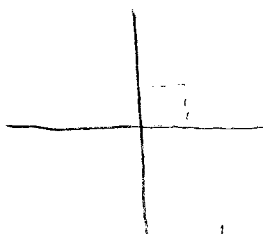
<u>Thm.</u> If a vector space V has a finite basis (generating & lin. indep) then every other basis of V is also finite and has the same number of elements.

<u>Corollary:</u> ~~This~~ If V has a finite basis, then it makes sense to define
$\dim V$ = the number of elements in a basis of V.
In this case, we say that V is "finite-dimensional"

<u>Examples</u>



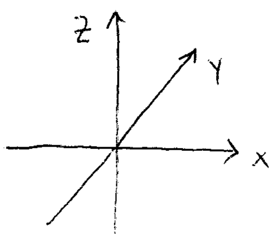
$= \mathbb{R}^2$

$\dim \mathbb{R}^2 = 2$

basis: $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -\pi \\ e \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$



$= \mathbb{R}^3$

basis: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathbb{R}^3 = 3$

~~$\mathbb{R}^4 =$~~ $\mathbb{R}^4 = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right\}$ basis: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathbb{R}^4 = 4$

$\dim \mathbb{R}^n = n$

<u>From Before:</u> If a finite set S generates a vector space V, then it has a subset $\beta \subset S$ which is a basis.

<u>Pf.</u> ~~If S is $\emptyset$, then $V = \{0\}$, take $\beta = \emptyset$ and it's a basis. Otherwise, take some $u_1 \in S$ s.t. $u_1 \neq 0$.~~

Pf. If $V = \{0\}$, take $\beta = \emptyset$ and it's a basis.

~~Otherwise take some $u_1 \in S$ s.t. $u_1 \neq 0$, take u_2~~

Otherwise V contains a non-zero vector, hence so does S,
so pick $u_1 \in S$ s.t. $u_1 \neq 0$

Pick $u_2 \in S$ s.t. $u_2 \notin \operatorname{span}(u_1)$

Pick $u_3 \in S$ s.t. $u_3 \notin \operatorname{span}(u_2, u_1)$

$\vdots$

Keep going until you cannot find some u_{k+1} in S s.t. $u_{k+1} \notin \operatorname{span}(u_1, \dots u_k)$.

The process is guaranteed to stop because S is finite.

At this stage, every $u \in S$ satisfies $u \in \operatorname{span}(u_1 \dots u_k)$

So $S \subset \operatorname{span}(u_1 \dots u_k)$

So $V = \operatorname{span} S \subset \operatorname{span}(u_1 \dots u_k)$

So if $\beta = \{u_1, \dots u_k\}$ then β generates V.

Claim: β is lin. indep, hence it is a basis.

Indeed, assume

$$a_1 u_1 + a_2 u_2 + \dots + a_k u_k = 0.$$

Furthermore, assume j is the maximal index for which $a_j \neq 0$. (if all $a_j = 0$, we're done).

So $\dfrac{a_1 u_1 + \dots a_j u_j}{a_j} = 0$

$$\frac{a_1}{a_j} u_1 + \dots \frac{a_j u_{j-1}}{a_{j-1}} + u_j = 0$$

$$\Rightarrow u_j = -\frac{a_1}{a_j} - \dots - \frac{a_{j-1}}{a_j} u_{j-1}$$

So, $u_j \in \operatorname{span}(u_1 \dots u_{j-1})$

$\Rightarrow\Leftarrow$

$\square$

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Lemma / **The Replacement Lemma**: Suppose a finite set G generates a v.s. V and assume $|G| = m$ ($\hookrightarrow$ # of elts)

$P(n)$ $\left\{\begin{array}{l} \text{Suppose } L \text{ is a lin. indep subset of } V \text{ and } |L| = n. \\ \text{Then } n \leq m \text{ and there exists a subset } H \subset G \text{ s.t.} \\ |H| = m-n \text{ and span } H \cup L = V \end{array}\right.$

Proof of the main thm, assuming the lemma: Let α and β be bases of V, and assume α is finite and $|\alpha| = m$. Take $G = \alpha$ and for some n, take $L = \{\beta_1 \ldots \beta_n\}$ the first n elts of β.
Then G generates as α is a basis, L is lin. indep, as $L \subset \beta$ & β is lin. indep., so by the lemma, $n \leq m$.
So β must be finite and $|\beta| \leq m$.
So $|\beta| \leq |\alpha|$. But now that we know that both α & β are finite, rerun the argument with α & β switching roles, and therefore $|\alpha| \leq |\beta|$
$\square$.

$$G = \{u_1, u_2, \ldots, u_{m-1}, u_m\} \qquad L = \{v_1, v_2, \ldots v_n\}$$

Pf By induction on n

Induction A statement $P(n)$ which depends on an integer n.

eg $\underbrace{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}_{P(n)}$

If you can prove that $P(n)$ implies $P(n+1)$ and you can prove that $P(0)$ holds, then $P(n)$ holds for all $n \geq 0$

Pf. By induction on n, with $P(n)$ as above.

Check $P(0)$: ~~[illegible]~~ If $n=0$, $|L|=0$, so $L=\emptyset$ and then
$0 \leq m$ (trivial)
and take $H=G$ and then $|H|=|G|=m=m-0=m-n$
and $\operatorname{span} H \cup L = \operatorname{span} G \cup \emptyset = \operatorname{span} G = V$

Now assume $n \geq 0$, and $P(n)$ holds.

without loss of generality

W.L.O.G, $G = \overbrace{\{u_1, \ldots u_m\}}^{m-n}$, $L = \{v_1 \ldots v_{n+1}\}$ and ~~then~~ the H we get for $L_1 = \{v_1 \ldots v_n\}$ is
$H_1 = \{u_1 \ldots u_{m-n}\}$

I.e., $n \leq m$ & $\operatorname{span}\{\underbrace{u_1 \ldots u_{m-n}}_{H_1}, \underbrace{v_1 \ldots v_n}_{L_1}\} = V$

So now $v_{n+1} \in V = \operatorname{span}\{u_1 \ldots u_{m-n}, v_1 \ldots v_n\}$

So $v_{n+1} = a_1 u_1 + a_2 u_2 + \ldots + a_{m-n} u_{m-n} + b_1 v_1 + \ldots + b_n v_n$.

If $a_1 = \ldots = a_{m-n} = 0$ then $v_{n+1} = \sum b_i v_i$

i.e, $-\sum b_i v_i + v_{n+1} = 0$ Contradicting the fact that L is linearly independent. So at least one $a_j \neq 0$.

W.L.O.G. $a_{m-n} \neq 0$

Then $a_{m-n}^{-1} v_{n+1} = \frac{a_1}{a_{m-n}} u_n + \ldots + \frac{a_{n-n-1}}{a_{m-n}} u_{m-n-1} + 1 \cdot u_{m-n} + \frac{b_1}{a_{m-n}} v_1 + \ldots + \frac{b_n}{a_{m-n}} v_n$

So $u_{m-n} = c_1 u_1 + c_2 u_2 + \ldots + c_{m-n-1} u_{m-n-1} + d_1 v_1 + \ldots + d_{n+1} v_{n+1}$

So u_{m-n} ~~belongs~~ $\in$ $\operatorname{span}\{u_1 \ldots u_{m-n-1}\} \cup \{v_1 \ldots v_{n-1}\}$.

So $\underset{V}{\operatorname{span}\{u_1 \ldots u_{m-n}, v_1 \ldots v_n\}} \subset \operatorname{span}\{u_1 \ldots u_{m-n-1}\} \cup \{v_1 \ldots v_{n+1}\}$

So $V = \operatorname{span}\{\underbrace{u_1 \ldots u_{m-n-1}}_{H}\} \cup \{v_1 \ldots v_{n+1}\}$

$\square$.

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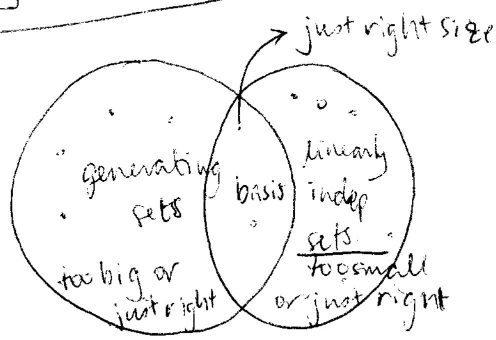
$P(n)$: $m \geq n$ & $\exists H \ldots |H| = m-n, \ldots$

$P(n+1)$: $m \geq n+1$ & $\exists H \ldots |H| = m-(n+1) = m-n-1$ ✓

The mere fact that we found H with $|H| = m-n-1$ implies $m-n-1 \geq 0 \Rightarrow m \geq n+1$.

□



Thm Let V be a v.s. and assume $\dim V = n$
Then 1. If span $G = V$ then $|G| \geq n$ and if $|G| = n$ then G is a basis.
2. If L is lin. indep, then $|L| \leq n$ and if $|L| = n$ then L is a basis.

Pf. 1. If span $G = V$ then there is $\beta \subset G$ which is a basis, so $|\beta| = n$ so $|G| \geq |\beta|$ so $|G| \geq n$.
If $|G| = n$, then $|G| = |\beta| \Rightarrow G = \beta$ so G is a basis.

2. Assume L is lin. indep, and take a basis β of V so $|\beta| = n$. Take $G = \beta$ and apply lemma: then

$$|L| \leq |G| = |\beta| = n$$

Furthermore, $\exists H \subset G$ s.t. $|H| = |G| - |L|$ (if $|L| = n$ then $|H| = |G| - |L| = n - n = 0$)
So $H = \emptyset$. and so that span $L \cup H$ = span G = span
If $|L| = n$, then $H = \emptyset$, so $L \cup \emptyset = L$ so span $L = V$ so L is generating $\Rightarrow L$ is a basis. □