MAT240 Oct. 5, 2006.

<u>General Fact:</u> $T \subset \text{span } S \Rightarrow \text{span } T \subset \text{span } S$

$(M_i\text{'s})$ $(N_i\text{'s})$

$\cap$ $\|$

lin comb of lin comb of elts in S $\subset$ lin comb of elts in S

$S \subset V$ is lin. indep $\Leftrightarrow$ whenever $u_i \in S$ are distinct

$\sum a_i u_i = 0 \Rightarrow \forall i\ a_i = 0$ "waste not"

<u>Comments</u> 1. $\emptyset$ is lin. indep

2. $\{u\}$ is lin indep. iff $u \neq 0$.

3. If $S_1 \subset S_2 \subset V$

a. If $S_1$ is linearly dependent, so is $S_2$

b. If $S_2$ is lin. indep, so is $S_1$.

c. If $S_1$ generates $(V)$ so does $S_2$

d. If $S_2$ doesn't generate $(V)$, neither do

4. If $S$ is lin indep in $V$ and $v \notin S$

then $S \cup \{v\}$ is lin. dep iff $v \in \text{span } S$.

<u>Pf.</u> $\Leftarrow$: Assume $v \in \text{span } S$

$v = \sum a_i u_i$ where $u_i \in S$

$\sum a_i u_i - 1 \cdot v = 0$

This is a lin. comb of elements in $S \cup \{v\}$, in which not all coefficients are 0, and which adds to 0.

So, $S \cup \{v\}$ is lin. dep.

$\Rightarrow$ Assume $S \cup \{v\}$ is lin. dep

$\Rightarrow$ Can find a lin comb

(*) $\sum a_i u_i + bv = 0$

where $u_i \in S$, not all of $a_i$'s and $b$ are 0.

(over)

If $b=0$, $\sum a_i u_i = 0$ & not all $a_i$'s are 0.

$\Rightarrow S'$ is lin. dep.

$\rightarrow$ Contradiction, so $b \neq 0$.

So divide by $b$: $(*)$ becomes

$$\sum \frac{a_i}{b} u_i + v = 0$$

$$\Rightarrow v = \sum \frac{-a_i}{b} u_i$$

$$\Rightarrow v \in \text{span } S'$$

$\square$.

<u>Def</u>: A basis of a v.s. $V$ is a subset $\beta \subset V$ s.t.

1. $\beta$ generates $V$ ($V = \text{span } \beta$)
2. $\beta$ is lin. indep

<u>Examples</u>

1. $\beta = \emptyset$ is a basis of $\{0\}$
2. $V$ be $\mathbb{R}$ as a v.s. over $\mathbb{R}$. $\beta = \{5\}$, $\beta = \{1\}$ are b.
3. $V$ be $\mathbb{C}$ as a v.s. over $\mathbb{R}$. $\beta = \{1, i\}$

   check: 1. $z = a + bi = a \cdot 1 + b \cdot i$

   So $\{1, i\}$ generate.

   2. Assume $a \cdot 1 + b \cdot i = 0$ where $a, b \in \mathbb{R}$.

   $\Rightarrow a + bi = 0$

   $\Rightarrow a = 0$, $b = 0$

4. $V = \mathbb{R}^n - \left\{\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}\right\}$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

   $e_1 \dots e_n$ are a basis of $V$.

   1. They span $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum a_i e_i$

   2. $\sum a_i e_i = 0$

   $\Rightarrow \sum a_i e_i = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$

   $\Rightarrow a_i = 0 \ \forall i$

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5. In $V = P_3(\mathbb{R})$ $\quad \beta = \{1, x, x^2, x^3\}$

6. In $P_1(\mathbb{R}) = \{ax+b\}$ $\quad \beta = \{\overset{u_1}{1+x}, \overset{u_2}{1-x}\}$ is a basis

1. $\frac{1}{2}(u_1+u_2) = 2 \times \frac{1}{2} = 1$ $\quad$ So $1 \in \operatorname{span}\beta$

$\frac{1}{2}(u_1 - u_2) = 2x = x$ $\quad$ So $x \in \operatorname{span}\beta$.

So $\underset{P_1(\mathbb{R})}{\underset{\parallel}{\operatorname{span}\{1,x\}}} \subset \operatorname{span}\beta$

2. Assume $au_1 + bu_2 = 0$

$\Rightarrow a(1+x) + b(1-x) = 0$

$(a+b) + (a-b)x = 0$

$\Rightarrow a+b=0$

$a-b=0$

$\overset{\text{sum}}{\Rightarrow} 2a = 0 \quad \overset{\text{dif}}{\Rightarrow} 2b = 0.$

$a = 0. \quad b = 0.$

$\therefore$ lin indep

<u>Thm.</u> A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed as a lin. comb. of elements in $\beta$ in exactly one way.

<u>Pf.</u> It's a combination of things we already know.