PSet 8: Partial Solutions

DISCLAIMER: I cannot claim that what I have written here constitutes a perfect solution. Certainly some mistakes are present; hopefully these mistakes aren't too severe. I hope that my answers may serve as a guide to you when studying for the final exam.

Problem 12.3

a)

We show that if f is integrable over $Q = A \times B$, then g is integrable over A, and $\int_Q f = \int_A g$. Let $\overline{I}(x) = \overline{\int}_{y\in B} f(x,y)$, and $\underline{I}(x) = \underline{\int}_{y\in B} f(x,y)$

Proof. By Problem 10.1 and by Theorem 10.3 in Munkres, we have the following relation:

$$\underline{\int}_{x\in A} \underline{I}(x) \leq \underline{\int}_{x\in A} g(x) \leq \overline{\int}_{x\in A} g(x) \leq \overline{\int}_{x\in A} \overline{I}(x). \tag{1}$$

By Fubini's Theorem, since f is integrable over Q, then both $\underline{I}$ and $\overline{I}$ are integrable over A, and:

$$\int_Q f = \int_{x\in A} \underline{I}(x) = \int_{x\in A} \overline{I}(x).$$

It follows that the leftmost and rightmost terms in equation (1) are each equal to $\int_Q f$, so that

$$\int_Q f = \underline{\int}_{x\in A} g(x) \leq \overline{\int}_{x\in A} g(x) = \int_Q f.$$

Hence, $\underline{\int}_{x\in A} g(x) = \overline{\int}_{x\in A} g(x)$, so that $\int_A g$ exists and is equal to $\int_Q f$, as required. □

b)

Let $A = B = [0,1]$, and consider the function $f : [0,1]^2 \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 1, & \text{if } y = \frac{1}{2} \text{ and } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

Step 1: We show that $\int_{x\in A}\int_{y\in B} f(x,y)$ exists. Fix x. Then $f(x,y)$ vanishes for all $y \in [0,1]$, except perhaps at the point $y = \frac{1}{2}$, if $x \in \mathbb{Q}$. In either case, $f(x,y)$ vanishes for $y \in B$ except on a *closed* set of measure zero: $\{\frac{1}{2}\}$ if $x \in \mathbb{Q}$, and $\varnothing$ otherwise. Thus, $\int_{y\in B} f(x,y)$ exists and equals 0 by Exercise 11.8 in Munkres, which was proved on a past problem set. Then $\int_{y\in B} f(x,y)$ is a constant function of x on A, and so $\int_{x\in A}\int_{y\in B} f(x,y)$ exists, as required.

Step 2: We show that $\int_{y\in B}\int_{x\in A} f(x,y)$ does not exist. Let $y=\frac{1}{2}$, and let $g(x)=f(x,\frac{1}{2})$. We claim that $\int_{x\in A} g(x)$ does not exist. For if P is any partition of $A=[0,1]$ and R is any rectangle determined by P, then, since the rationals and irrationals are dense in $[0,1]$, $m_R(g)=0$ and $M_R(g)=1$. Hence,

$$L(g,P)=\sum_R m_R(g)v(R)=\sum_R 0\cdot v(R)=0,$$

and

$$U(g,P)=\sum_R M_R(g)v(R)=\sum_R 1\cdot v(R)=1.$$

Since this holds for any partition P of A, then the upper and lower sums cannot be made arbitrarily close. Hence, $\int_{x\in A} g(x)=\int_{x\in A} f(x,\frac{1}{2})$ does not exist. Moreover, it follows that $\int_{y\in B}\int_{x\in A} f(x,y)$ does not exist.

Step 3: We show that $\int_Q f$ exists. We claim that f is continuous everywhere except on the line with $y=\frac{1}{2}$, or more formally, that its set of discontinuities is $D=\{(x,\frac{1}{2}) : x\in[0,1]\}$. Take a point not in D. Since the line D is closed in $[0,1]^2$, then for any point $p\notin D$ we may choose $\delta>0$ such that $B(p;\delta)\cap D=\varnothing$. Moreover, that f is discontinuous on D follows from the argument in Step 2. It remains to be shown that D is of measure zero, from which it follows by Theorem 11.2 that f is integrable on $[0,1]^2$. Given any $\epsilon>0$, let $Q_\epsilon=[0,1]\times[\frac{1}{2}-\epsilon/4,\frac{1}{2}+\epsilon/4]$. Then Q_ϵ is a covering by countably many rectangles which satisfies $v(Q_\epsilon)=\epsilon/2<\epsilon$.

c)

Let $A=B=[0,1]$. For each positive integer k, let $S_k=\{\frac{m}{2^k} : m\in\mathbb{N}\cap[1,2^k-1]$ and m is odd$\}$, let $S\subset A\times B$ be defined as $S=\cup_k^\infty(S_k\times S_k)$, and define $f_S : A\times B\to\mathbb{R}$ by

$$f_S(x,y)=\begin{cases}1, & \text{if } (x,y)\in S\\ 0, & \text{otherwise}\end{cases}$$

It is clear that f_S is bounded. This can be visualized by marking a point at the center of the unit square, then dividing this square into 4 equal squares and marking the points at the centers of each of these squares, and proceeding recursively on the 4 sub-squares of each of these squares.

Step 1: We show that $\int_{x\in A}\int_{y\in B} f_S(x,y)$ exists. Fix $x_0\notin S$. Then $f_S(x_0,y)$ is identically zero for $y\in B=[0,1]$, so that $\int_{y\in B} f_S(x_0,y)=0$. If we instead choose to fix $x_1\in S$, so that x_1 may written in lowest terms as $x_1=\frac{m}{2^k}$ for some positive integer m, then there exist at most finitely many points y, i.e. for $y\in Y=\{1,3,5,...,2^k-1\}$, at which $f_S(x_1,y)=1\neq 0$. Note that since Y is a finite collection, it is closed set of measure zero in $[0,1]$. Hence, we may evoke exercise 11.8 in Munkres to conclude that $\int_{y\in B} f_S(x_1,y)$ exists and that $\int_{y\in B} f_S(x_1,y)=0$. It follows that $\int_{y\in B} f_S$ is identically zero for all $x\in A=[0,1]$, and so $\int_{x\in A}\int_{y\in B} f_S(x,y)=0$.

Step 2: That $\int_{y\in B}\int_{x\in A} f_S(x,y)$ exists can be demonstrated by an analogous argument.

Step 3: We claim that f_S is not integrable over $A\times B=[0,1]^2$. Let P be any partition of $[0,1]^2$. Then P can be expressed as $P=(P_A,P_B)$, where P_A is a partition of A and P_B is a partition of B. Consider any rectangle $R_A=[a,\alpha]\subset A$ determined by the partition P_A, and any rectangle $R_B=[b,\beta]\subset B$ determined by the partition P_B. Now, choose $N\in\mathbb{N}$ large enough that $\frac{1}{2^N}<\min(\alpha-a,\beta-b)$. Then there exists a pair of points of the form $\frac{m}{2^N},\frac{m+1}{2^N}$ contained in the rectangle R_A, where m is some positive integer. Let m' equal whichever of m or $m+1$ is odd, and let $x_0=\frac{m'}{2^N}\in R_A$. In an analogous manner, find a point $y_0=\frac{n'}{2^N}\in R_B$ where n' is some odd positive integer. Then $x_0,y_0\in S_N$, so that the point (x_0,y_0) is in our set S. It follows that $M_{R_A\times R_B}(f_S)=1$. But clearly $m_{R_A\times R_B}(f_S)=0$. Hence,

$$L(f_S,P)=\sum_{R_A\times R_B} 0\cdot V(R_A\times R_B)=0$$

and

$$U(f_S,P)=\sum_{R_A\times R_B} 1\cdot V(R_A\times R_B)=V([0,1]^2)=1.$$

Since the partition P was chosen arbitrarily, it follows that $\underline{\int}_{A\times B} f_S=0\neq 1=\overline{\int}_{A\times B} f_S$, and so f_S is not integrable over $A\times B$.

Problem A

Let $Q=[0,1]^3$, and $f:Q\to\mathbb{R}$ be a bounded and given by $f(x,y,z)=1$ when $x<y<z$, and $f(x,y,z)=0$ otherwise. We claim that $\int_Q f=\frac{1}{6}$.

Proof. By Fubini's Theorem, we may write:

$$\int_Q f=\int_{(y,z)\in[0,1]^2}\underline{\int}_{x\in[0,1]} f(x,y,z).$$

We note that $\int_{x\in[0,1]} f(x,y,z)$ exists for any fixed $(y,z)\in[0,1]^2$; for if $y\geq z$, the function is zero for all $x\in[0,1]$, and if $y<z$, the function is discontinuous only at the point $x=y$, and hence only on a set of measure zero. Therefore,

$$\int_Q f=\int_{(y,z)\in[0,1]^2}\int_{x\in[0,1]} f(x,y,z)$$

If we assume that $y<z$, then $f(x,y,z)=1$ for all $x\in[0,y)$ and so we have:

$$\begin{aligned}\int_Q f&=\int_{(y,z)\in[0,z]\times[0,1]}\int_{x\in[0,y]} 1\\&=\int_{(y,z)\in[0,z]\times[0,1]} y\end{aligned}$$

Since the function $g(y) = y$ is continuous on $[0, z] \times [0, 1]$, then by Corollary 12.4,

$$\begin{aligned}\int_Q f &= \int_{z\in[0,1]} \int_{y\in[0,z]} y \\ &= \int_{z\in[0,1]} \frac{z^2}{2} \\ &= \frac{1}{6}\end{aligned}$$

□