Lecture 14.10.06

Lemma If P, Q are invertible

$$\begin{tikzcd} V \arrow[loop left, "Q"] \arrow[r, "T"] & W \arrow[loop right, "P"] \end{tikzcd}$$

$$\operatorname{rank} T = \operatorname{rank} PT = \operatorname{rank} TQ = \operatorname{rank} PTQ$$

Proof: $\operatorname{rank} T = \operatorname{rank} TQ$: Q is is onto, therefore

range

$$\operatorname{ran} T = \operatorname{ran} TQ$$

$\operatorname{rank} PT = \operatorname{rank} T$

$\operatorname{ran} PT = P(\operatorname{ran} T)$

If P is 1-1 & V is a subspace, then

$\dim V = \dim P V$

So $\operatorname{rank} PT = \operatorname{rank} T$

$\operatorname{rank} T = \operatorname{rank} PT = \operatorname{rank} PTQ$ (can multiply Q from the right)

Principle (to be revisited later): Changing a basis is multiplying by an invertible matrix / linear transformations

$$V \xrightarrow{T} W \qquad A = [T]_\beta^\gamma \qquad B = [T]_{\beta'}^{\gamma'}$$

$\beta \to \beta'$ $\qquad \gamma \to \gamma'$ $\qquad B = PAQ$ for some invertible P, Q.

Corollary: $\operatorname{rank} A$ is well defined.

<u>THM:</u> Any matrix A can be row & column reduced to a "block" matrix of the form $\left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right) \Rightarrow \operatorname{rank} A = r$

<u>Proof:</u> If $A = 0$ then it is of that form $\left(0 \right)$

otherwise it has a non-zero entry somewhere: $\begin{pmatrix} & * \end{pmatrix}$ by row & column swaps bring the non-zero entry to top left.

Divide the first row by that entry & our matrix becomes $\begin{pmatrix} 1 & \cdots \\ \vdots & \ddots \end{pmatrix}$

using row ops you can kill everything on the first column, other than the 1. Using column ops, do same to first row. $\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$

by induction B can be reduced to $B' = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$

doing the same row/column ops to the bigger matrix we get $\left(\begin{array}{c|c|c} 1 & 0\,0\,0 & 0 \\ 0 & I_{p\times p} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

$= \left(\begin{array}{c|c} I_{(p+1)(p+1)} & 0 \\ \hline 0 & 0 \end{array}\right)$

<u>Recall</u> If $A \in M_{m\times n}(F)$ then $A^T \in M_{n\times m}(F)$

$A = \begin{pmatrix} d & r & o & r \\ m & a & x & i \\ s & t & a & n \end{pmatrix}$ $\quad A^T = \begin{pmatrix} d & m & s \\ r & a & t \\ o & x & a \\ r & i & n \end{pmatrix}$ $\quad (A^T)_{ij} = A_{ji}$

Claim: $\operatorname{rank} A = \operatorname{rank} A^T$

Proof:

$$\begin{pmatrix} \text{dror} \\ \text{maxi} \\ \text{stan} \end{pmatrix} \xrightarrow{\text{row \& col. ops}} \left(\begin{array}{c|c} \overset{\text{square}}{I_{r\times r}} & \overset{\text{rectangular}}{\boxed{0}} \\ \hline \boxed{0} & 0 \end{array}\right) \quad \text{rank} = r$$

$$\begin{pmatrix} d & m & s \\ r & a & t \\ o & x & a \\ r & i & n \end{pmatrix} \xrightarrow{\text{col \& row ops}} \left(\begin{array}{c|c} I_{r\times r} & \boxed{0} \\ \hline \boxed{0} & \boxed{0} \end{array}\right) \quad \text{rank} = r$$

Def: Let $A \in M_{m\times n}(F)$

$$A = \underbrace{\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix}}_{n} \Big\} m \qquad \text{col-space } A = \operatorname{span}(c_1, \dots, c_n) \subset F^m$$

$c_1, \dots, c_n \in F^m$

$$A = \begin{pmatrix} \text{—— } r_1 \text{ ——} \\ \text{—— } r_2 \text{ ——} \\ \vdots \\ \text{—— } r_n \text{ ——} \end{pmatrix} \qquad \text{row-space } A = \operatorname{span}(r_1, \dots, r_m) \subset F^n$$

$r_1, \dots, r_m \in F^n$

Claim: $\dim \text{col-space}(A) = \dim \text{row-space}(A) = \operatorname{rank} A$

Proof: A defines a linear trans $T: F^n \longrightarrow F^m$ by mapping $\underset{F^n = M_{n\times 1}}{v} \longmapsto \underset{M_{m\times 1} = F^m}{A.v}$

I.e. $T(v) = A\cdot v$ $\quad m\Big\{\underbrace{\begin{pmatrix} \quad \end{pmatrix}}_{n} \begin{pmatrix} v \end{pmatrix} = \begin{pmatrix} Av \end{pmatrix}$

It is easy to check that if (e_i) is the std basis of $\mathbb{R}^m$ & (F_j) is the std basis of $\mathbb{R}^n$, then

$$\underbrace{\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix}}_{A} \cdot f_j = c_j$$

$$[T]_{(f_j)}^{(e_i)} = A$$

Pf: $T(f_j) = A \cdot f_j = c_j$

$$T \mapsto \begin{pmatrix} | \\ c_j \\ | \end{pmatrix}$$

1.

rank A = rank T = dim ran(T) = dim span(Tf_j) = dim span (c_j) = dim col. space

$$\begin{pmatrix} | & | \end{pmatrix} = \begin{pmatrix} | & | & & \\ c_1 & c_2 & \cdots & \\ | & | & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ choses } c_1, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ choses } c_2, \ldots$$

This proves rank A = dim(col-space(A))

yet

rank A = rank A^T

= dim(colsspace (A^T))

= dim (row-space(A))

Q: How far can you go whith row operations alone?

A: You can bring A to "reduced row-echelon form"

Q. Reduced row echelon form?

A: It is as far as you can get with row-ops.

1. All zero row are at the bottom.

2. In every non-zero row the leading entry is 1.

3. In the column of any such 1, all other entries are 0.

4. The leading 1's are in "echelon form".

Q: What's the rank of a matrix in r.r.ech. form?

A: It is the number of non-zero rows

Q: If A is invertible (nxn) & A' is the result of row reducing to A.

(A' is in r.r.e.f) What's A'?

A: rank $A = n$ in A' there are n leading 1's.

$$A' = \underbrace{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}}_{n} \Big\} n \Rightarrow A' = I$$

$E_p \cdots E_3 E_2 E_1 A = A' = I$ where $E_1 \cdots E_p$ are the elementary matrices

$$\underbrace{E_p \cdots E_3 E_2 E_1}_{\overset{\shortparallel}{A^{-1}}} (A) = I$$

"The inverse of A is the product of the elementary matrices used to row reduce it to r.r.e.f.

$$A^{-1} = \left(E_p E_{p-1} \cdots E_1\right) \cdot I \quad \text{mutiply by } I$$

" A^{-1} is the result of applying to I all the row ops you would have applied to A to get it to r.r.e.f.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \left.\vphantom{\Big|}\right\} \text{row ops} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left.\vphantom{\Big|}\right\} \text{same row ops} \quad A^{-1}$$

Let B be the block matrix $B = \left(A \mid I\right)$

do row ops to B to make left half $= I$

now read A^{-1} off the right half.

Example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad B = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$

$$\mapsto \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \underset{\text{add } ② \cdot (-2) + ①}{\longmapsto} \left(\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} -2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$