$$\det\left(I_{n\times n}\right) = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \end{pmatrix} = 1\cdot\det I_{(n-1)\times(n-1)} - 0\cdot\det\left(\cdots\right) + 0\det\left(\ \right) - \cdots$$

$$= \det I_{(n-1)\times(n-1)} = \det I_{(n-2)\times(n-2)} = \cdots = \det(I)_{1\times 1} = \det(1) = 1$$

23.11.06

THM 2: A is invertible $\iff$ $|A|$ is invertible

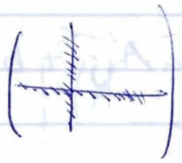
Thm 1: det is fully computable using row ops

1. $\det E'_{i,j} A = -\det A \qquad \det E'_{j,j} = -1$

2. $\det E^2_{i,j} A = c\det A \qquad \det E^2_{i,c} = C$

3. $\det E^3_{i,j} = \det A \qquad \det E^3_{i,j,c} = 1$

$$\det A = |A| = \sum_{j=1}^{n} (-1)^{1+j} A_{1j}\, |\tilde{A}_{1j}| \quad ; \quad \tilde{A}_{ij}$$



"Axiomatic Properties:

1. $\det I = 1$

2. Multilinearity: $\begin{vmatrix} \text{---} \\ ar' + br'' \\ \text{---} \end{vmatrix} = a\begin{vmatrix} \text{---} \\ r' \\ \text{---} \end{vmatrix} + b\begin{vmatrix} \text{---} \\ r'' \\ \text{---} \end{vmatrix}$

3. If A has two adjacent equal rows, $\det A = 0$

$$\begin{vmatrix} \text{---} \\ \text{--- } r \text{ ---} \\ \text{--- } r \text{ ---} \\ \text{---} \end{vmatrix} = 0$$

Proof of prop 2

If the row in question is the top row, obvious

$$\underbrace{\begin{vmatrix} ar'+br'' \\ \hline \\ \hline \\ \hline \end{vmatrix}}_{A} \overset{?}{=} a \underbrace{\begin{vmatrix} r' \\ \hline \\ \hline \\ \hline \end{vmatrix}}_{A'} + b \underbrace{\begin{vmatrix} r'' \\ \hline \\ \hline \\ \hline \end{vmatrix}}_{A''}$$

$$A_{ij} = \begin{cases} A'_{ij} = A''_{ij} & i>1 \\ aA'_{1j} + bA'_{1j} & i=1 \end{cases}$$

$$|A| = \sum (-1)^{1+j} A_{1j} |\tilde{A}_{1j}|$$

$$= \sum (-1)^{1+j} (aA'_{1j} + bA''_{1j}) |\tilde{A}_{1j}|$$

$$= a \sum (-1)^{1+j} A'_{1j} |\tilde{A}'_{1j}| + b \sum (-1)^{1+j} A''_{1j} |\tilde{A}'_{1j}|$$

$$= a|A'| + b|A''|$$

Otherwise det A is a l.c. of det $\tilde{A}_{ij}$ and in each $\tilde{A}_{1j}$, the row in question is bumped up once

Proof of Prop. 3

Without loss of generality the two rows are at the top.

$$\left(\text{Induction} \quad \underset{r_7 = r_8}{|A|} = \sum \cdots |\tilde{A}_{ij}| = 0 \right)$$

$$A = \begin{pmatrix} — r — \\ — r — \\ \square \end{pmatrix}$$

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} |\tilde{A}_{1j}|$$

$$= \sum_{j=1}^{n} (-1)^{1+j} \cdot A_{1j} \sum_{k=1}^{n-1} (-1)^{1+k} (\tilde{A}_{1j}) \cdot \left|\widetilde{(\tilde{A}_{1j})_k}\right|$$

$\tilde{\ }$ means remove

$\sum_{k=1}^{n-1} \rightarrow \sum_{l=1,\ l \neq j}^{n}$

$$= \sum_{\substack{j=1 \\ l=1 \\ j \neq l}}^{n} (-1)^{1+j} A_{1j} (-1)^{1+k} A_{2l} \left|\widetilde{A_{12,jl}}\right|$$

A with rows 1-2 & columns j,l removed

if $\ell < j \Rightarrow \sum_{\substack{j,\ell=1 \\ \ell<j}}^{n} (-1)^{1+j} A_{1j} (-1)^{1+\ell} A_{2\ell} |\widetilde{A_{1,12,j\ell}}|$

if $\ell > j$

$$+ \sum_{\ell>j} (-1)^{1+j} A_{1j} (-1)^{1+(\ell-1)} A_{2\ell} |\tilde{A}_{12,j\ell}|$$

$$= \sum_{\ell<j} (-1)^{j+\ell} r_j r_\ell \left| \widetilde{A_{12,j\ell}} \right|$$

$$+ \sum_{j>\ell} (-1)^{1+\ell} (-1)^{j} \underset{\substack{\downarrow \\ r_\ell}}{A_{1\ell}} \underset{\substack{\downarrow \\ r_j}}{A_{2j}} |\tilde{A}_{12,\ell j}| \quad \text{(swap } j \sim \ell \text{ above)}$$

$$= \sum_{\ell<j} (-1)^{j+\ell} r_j r_\ell |\widetilde{A_{12,j\ell}}| + \sum_{j>\ell} (-1)^{1+\ell} (-1)^{j} r_\ell r_j |\widetilde{A_{12,\ell j}}|$$

= first term − first term = 0