Handout on the Frobenius method for MAT267

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Keep in mind what it means to be analytic: a function f being analytic in a neighborhood of $x = 0$ means that it is equal to a power series in a neighborhood of $x = 0$.

We have the second order ODE

$$x^2 y''(x) + xp(x)y'(x) + q(x)y(x) = 0,$$

where $p(x) = \sum_{n=0}^{\infty} p_n x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n x^n$ are analytic in a neighborhood of $x = 0$. Let $I(r) = r(r-1) + p_0 r + q_0$, the indicial polynomial, and say that the roots are α_1, α_2 and $\alpha_1 = \alpha_2 + n$ for some positive integer n. We have one solution $y_1(x) = x^{\alpha_1} f(x)$, i.e. in the form of a Frobenius series, where f is analytic in a neighborhood of $x = 0$, and we wish to obtain a second independent solution.

We shall see what conditions C would have to satisfy for there to be a solution of the form $y_2(x) = C(x)y_1(x)$. The way we play this game is by assuming y_2 is a solution, putting $y_2 = Cy_1$ into the equation, and then getting an explicit formula for C from this. Once we have the explicit formula for C one can then verify that we can determine its Taylor coefficients to make it solve the equation; that is routine and we don't show that here, although it is very easy to make a mistake.

Putting $y_2 = Cy_1$ into the ODE we get (remember that y_1 is not just an arbitrary function but solves the ODE)

$$\frac{C''(x)}{C'(x)} + 2\frac{y_1'(x)}{y_1(x)} + \frac{p(x)}{x} = 0.$$

Then let a be a constant, (there are certain constraints on what a can be)

$$\log C'(x) + 2\log y_1(x) + \int_a^x \frac{p(t)}{t} dt = A$$

where A is the constant we get by evaluating the indefinite integrals at the left endpoint. But

$$\int_a^x \frac{p(t)}{t} dt = \int_a^x \frac{p_0}{t} + \sum_{n=1}^{\infty} p_n t^{n-1} dt = p_0 x + F(x),$$

where F is some function that is analytic in a neighborhood of $x = 0$. Let $G(x) = A - F(x)$, and then we get

$$\log C'(x) + 2\log y_1(x) + \log(x^{p_0}) = G(x),$$

giving

$$C'(x) y_1(x)^2 x^{p_0} = e^{G(x)}.$$

Therefore, as $y(x) = x^{\alpha_1} f(x)$,

$$C(x) = \int_a^x \frac{e^{G(t)}}{y_1(t)^2 t^{p_0}} dt = \int_a^x \frac{e^{G(t)}}{t^{2\alpha_1} f(t)^2 t^{p_0}} dt = \int_a^x \frac{H(t)}{t^{2\alpha_1 + p_0}} dt,$$

and H is analytic at $x = 0$ because by assumption $f(0) \neq 0$ (we assumed that the constant coefficient in f was nonzero). Note that because α_1, α_2 are the roots of the polynomial $r(r-1) + p_0 r + q_0$ we have $-\alpha_1 - \alpha_2 = p_0 - 1$. Since $\alpha_1 - \alpha_2 = n$, we have

$$C(x) = \int_a^x \frac{H(t)}{t^{n+1}} dt.$$

Then

$$C(x) = \int_a^x \sum_{j=0}^{n-1} h_j \frac{t^j}{t^{n+1}} + \frac{h_j t^n}{t^{n+1}} + \sum_{j=n+1}^{\infty} h_j \frac{t^j}{t^{n+1}}$$

which is

$$C(x) = \int_a^x \sum_{j=0}^{n-1} h_n t^{j-n-1} + \frac{h_n}{t} + \sum_{j=n+1}^{\infty} h_j t^{j-n-1} dt$$

which is

$$C(x) = \sum_{j=0}^{n-1} h_j \frac{x^{j-n}}{j-n} + h_n \log x + K_1(x)$$

where K_1 is analytic at $x = 0$ (we had to evaluate the indefinite integrals at $t = a$, and what we got from doing that we put into K_1, as well as what we got from the series going from $j = n+1$ to ∞). This is equal to

$$C(x) = x^{-n} \sum_{j=0}^{n-1} h_j \frac{x^j}{j-n} + h_n \log x + x^{-n} x^n K_1(x).$$

Take $K_2(x) = \sum_{j=0}^{n-1} h_j \frac{x^j}{j-n} + x^n K_1(x)$, which is itself analytic at $x = 0$. So we have

$$C(x) = x^{-n} K_2(x) + h_n \log x.$$

Thus our second solution is, remembering that $\alpha_1 = \alpha_2 + n$,

$$y_2(x) = C(x) y_1(x) = \left(x^{-n} K_2(x) + h_n \log x\right) y_1(x) = \left(x^{-n} K_2(x) + h_n \log x\right) x^{\alpha_1} f(x)$$

which is

$$y_2(x) = x^{\alpha_2} K_3(x) + h_n y_1(x) \log x.$$