for each pair $a, b \in F$, and each $v_1 \in V$, $w_1 \in W$

$(ab)v_1 = a(bv_1)$ $\quad (ab)w_1 = a(bw_1)$

For Z: for each $(v_1, w_1) \in Z$

$$\begin{aligned}(ab)(v_1, w_1) &= ((ab)v_1, (ab)w_1) \\ &= (a(bv_1), a(bw_1)) \\ &= a(bv_1, bw_1) \\ &= a(b(v_1, w_1))\end{aligned}$$

Hence Z follow VS6.

⑧ For VS7 (For each element $a \in F$, and each pair $x, y \in V$, $a(x+y) = ax+ay$.)

$\because$ V, W are vector spaces over F

$\therefore$ By VS7, In V, W.

for each $a \in F$, $v_1, v_2 \in V$, $w_1, w_2 \in W$

$a(v_1+v_2) = av_1 + av_2$ $\quad a(w_1+w_2) = aw_1 + aw_2$

For Z: for each $(v_1, w_1), (v_2, w_2) \in Z$

$$\begin{aligned}a[(v_1, w_1) + (v_2, w_2)] &= a(v_1+v_2, w_1+w_2) \\ &= (a(v_1+v_2), a(w_1, w_2)) \\ &= (av_1+av_2, aw_1+aw_2) \\ &= (av_1, aw_1) + (av_2, aw_2) \\ &= a(v_1, w_1) + a(v_2, w_2)\end{aligned}$$

Hence Z follows VS 7

⑨ For VS8 (For each $a, b \in F$, $x \in V$
$(a+b)x = ax + bx$)
∵ V, W are vector spaces over F
∴ By VS8, In V and W
for each $a, b \in F$, $v_1 \in V$, $w_1 \in W$
$(a+b)v_1 = av_1 + bv_1$ $(a+b)w_1 = aw_1 + bw_1$

For Z: for each $(v_1, w_1) \in Z$

$$\begin{aligned}(a+b)(v_1, w_1) &= ((a+b)v_1, (a+b)w_1) \\ &= (av_1 + bv_1, aw_1 + bw_1) \\ &= (av_1, aw_1) + (bv_1, bw_1) \\ &= a(v_1, w_1) + b(v_1, w_1)\end{aligned}$$

Hence Z follows VS8.

From above all statements.

Z over a field F is a set with a special element 0_Z ($0_Z = (0_V, 0_W)$) and two binary operations:

$+ : Z \times Z \to Z$
$\times : F \times Z \to Z$

and Z follows VS1-8.

Hence Z is a vector space.

P30

8. Solution:

a) Yes $W_1 = \{(3a_2, a_2, -a_2) \in R^3\}$

let $a_2 = 0 \Rightarrow (0,0,0) \in R^3 \Rightarrow (0,0,0) \in W_1$

Assume a_2' s.t. $(3a_2', a_2', -a_2') \in R^3$

$\because (3a_2, a_2, -a_2) + (3a_2', a_2', -a_2')$

$= (3a_2 + 3a_2', a_2 + a_2', -a_2 - a_2') \in R^3$

$3a_2 + 3a_2' = 3(a_2 + a_2')$, $-a_2 - a_2' = -(a_2 + a_2')$

$\therefore (3a_2 + 3a_2', a_2 + a_2', -a_2 - a_2') \in W_1$

W_1 is closed under addition

$\because a(3a_2, a_2, -a_2) = (3aa_2, aa_2, -aa_2) \in R^3$ $3aa_2 = 3(aa_2)$ $-aa_2 = -(aa_2)$

$\therefore (3aa_2, aa_2, -aa_2) \in W_1$ $\therefore W_1$ is a subspace of R^3

W_1 is closed under multiplication

b) No.

$W_2 = \{(a_3 + 2, a_2, a_3) \in R^3\}$

$\because a_3 + 2 \neq a_3$ $\therefore a_3 + 2 = 0$ and $a_3 = 0$ cannot be exosted together

$\therefore 0 \notin W_2$. $\therefore W_2$ is not a subspace of R^3.

c) Yes

let $a_1 = a_2 = a_3 = 0$, $2a_1 - 7a_2 + a_3 = 0 - 0 + 0 = 0$.

$(0,0,0) \in R^3 \Rightarrow (0,0,0) \in W_3$.

$a_2 = \frac{2a_1 + a_3}{7}$

Assume. a_1', a_3' s.t. $(a_1', \frac{2a_1' + a_3'}{7}, a_3') \in R^3$.

$\because (a_1', \frac{2a_1' + a_3'}{7}, a_3') + (a_1, \frac{2a_1 + a_3}{7}, a_3)$

$= (a_1 + a_1', \frac{2a_1 + 2a_1' + a_3 + a_3'}{7}, a_3 + a_3') \in R^3$. $\frac{2a_1 + 2a_1' + a_3 + a_3'}{7} = \frac{2(a_1 + a_1') + (a_3 + a_3')}{7}$

$\therefore (a_1 + a_1', \frac{2a_1 + 2a_1' + a_3 + a_3'}{7}, a_3 + a_3') \in W_3$.