MAT240 – Abstract Linear Algebra Lecture

The real numbers A set **R** with two binary operations (+, *) and two special elements 0, 1 ∈ **R** s.t.:

R1. $\forall a, b \in R \quad a + b = b + a \quad ab = ba$

R2. $\forall a, b, c \in R \quad (a + b)c = a + (b + c) \quad (ab)c = a(bc)$

R3. $\forall a \in R \quad a + 0 = a \quad a * 1 = a$

R4. $\forall a \in R, \exists b \; s.t. \quad a + b = 0 \quad \forall a \neq 0 \in R, \exists b \; s.t. \quad a * b = 1$

R5. $\forall a, b, c \in R \quad (a + b)c = ac + bc$

$\quad \rightarrow (a + b)(a - b) = a^2 - b^2$

Definition: A file is a set F with two binary operations ($+: FxF \rightarrow F, *: FxF \rightarrow F$) and also two elements 0,1 ∈ F s.t.:

F1. *Commutativity* $\quad \forall a, b \in F \quad a + b = b + a \quad ab = ba$

F2. *Associativity* $\quad \forall a, b, c \in F \quad (a + b)c = a + (b + c) \quad (ab)c = a(bc)$

F3. $\forall a \in F \quad a + 0 = a \quad a * 1 = a$

F4. $\forall a \in V, \exists b \; s.t. \quad a + b = 0 \quad \forall a \neq 0 \in F, \exists b \; s.t. \quad a * b = 1$

F5. *Distributivity* $\forall a, b, c \in F \quad (a + b)c = ac + bc$

Examples:

1. $F = R$
2. $F = Q = \{\frac{p}{q} \; s.t. \, p, q \in Z\}$
3. $C = \{a + bi: \; a, b \in R\}$
4. $F_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
0	0	1

5. $F_7 = \{0, 1, 2, 3, 4, 5, 6\}$

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Note: there exists a multiplicative inverse for this 'field'. Therefore, it is a field.

6. $F_6 = \{0, 1, 2, 3, 4, 5\}$

k	0	1	2	3	4	5
2k	0	2	4	0	2	4

There is no 1 value, therefore no multiplicative inverse exists. Therefore, it not a field!

Theorem: F_p for p>1 is a field iff P is a prime.

Tedious Theorems:

1. $a + b = c + b \rightarrow a = c$ *"Cancellation Property"*

    Proof: $a + b = c + b$

    $(a + b) + d = (c + b) + d$ (by property F4 there exists d such that b+d=0)

    $a + (b + d) = c + (b + d)$

    $a + 0 = c + 0$

    $a = c$ (by property F3)

2. $a * b = c * b \quad if \ b \neq 0 \rightarrow a = c$
3. $If \ a + 0` = a \rightarrow 0` = 0$
4. $If \ a * 1` = a \ \ and \ a \neq 0 \rightarrow 1` = 1$
5. $a + b = 0 = a + b` \rightarrow b = b`$
6. $ab = 1 = ab` \rightarrow b = b` \rightarrow a^{-1} \ makes \ sense \ iff \ a \neq 0$
7. $-(-a) = a$
8. $a * 0 = 0$

Note: This proof is particularly important because it mixes addition and multiplication. Moreover, any proof must use the distributive law.

Proof:

$$0 = a * (0 + 0)$$
$$a * 0 + a * 0 \qquad \text{(by F5)}$$
$$0 = a * 0$$

9. $\nexists 0^{-1}: \ \forall b, \ 0b \neq 1$
10. $(-a)b = a(-b) = -(ab)$
11. $(-a)(-b) = ab$
12. $a + b)(a - b) = a^2 - b^2$