Local Differentials and Matrix Factorizations

Dror Bar–Natan at UIUC, March 11, 2004, http://www.math.toronto.edu/~drorbn/Talks/UIUC–050311/

Quantum algebra:

Claim. If $ba=qab$ then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}$$

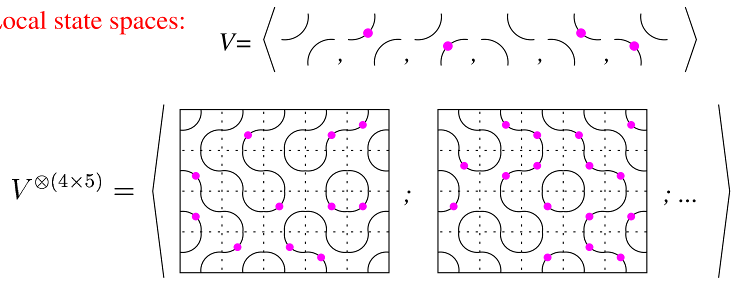
where

$$(n)_q := 1+q+\ldots+q^{n-1},$$
$$(n)!_q := (1)_q(2)_q\cdots(n)_q,$$
$$\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}.$$

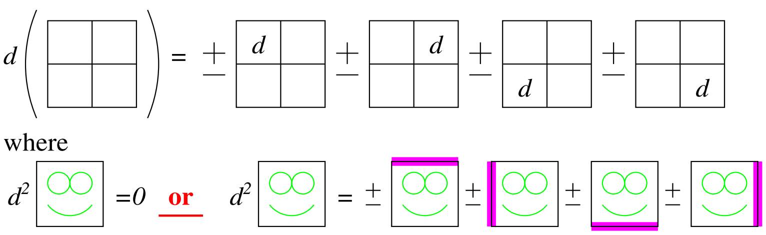
Conjecture: (I. Frenkel, though he may disown this version)

1. Every object in mathematics is the Euler characteristic of a complex.
2. Every operation in mathematics lifts to an operation between complexes.
3. Every identity in mathematics is true up to homotopy at complex–level.

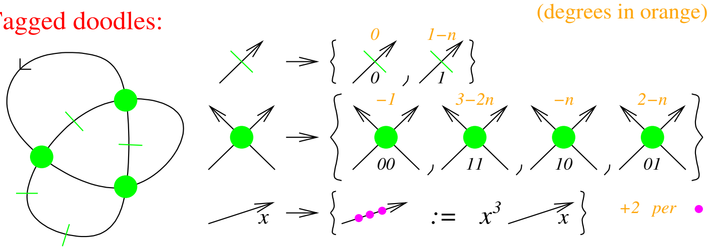
Local state spaces:

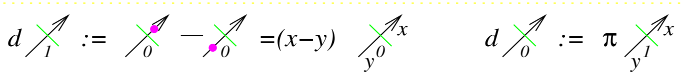


Local differentials:

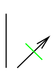


Tagged doodles: (degrees in orange)





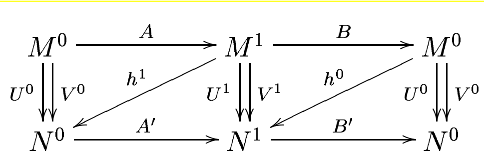

```
In[1]:= n = 2; π_{i_, j_} := Cancel[(x_i^(n+1) - x_j^(n+1))/(x_i - x_j)]; π_{1,2}
Out[1]= x1^2 + x1 x2 + x2^2
In[2]:= L = ( 0      x1 - x2 )
            ( π_{1,2}  0     );
        Expand[L.L] // MatrixForm
Out[3]//MatrixForm=
( x1^3 - x2^3       0       )
(      0       x1^3 - x2^3  )
```

Set $L=d$ 

(deg $d = n+1$)

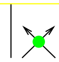
Matrix factorizations:

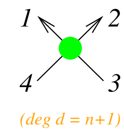
$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$
$$AB = BA = \omega I$$



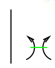
A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

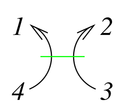
See Khovanov and Rozansky, arXiv:math.QA/0401268

Likewise, set $Q=d$  with:



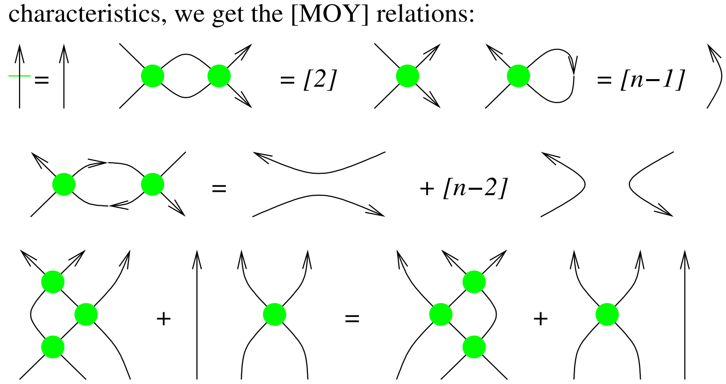

```
In[4]:= Q := ( 0    0    v1   v2  )
             ( 0    0    u2   -u1 )
             ( u1   v2   0    0   )
             ( u2   -v1  0    0   );
        {v1, v2} = {x1 + x2 - x3 - x4, x1 x2 - x3 x4};
In[6]:= g[s_, p_] :=
          s^(n+1) + (n+1) Sum[(-1)^i/i Binomial[n - i, i - 1] s^(n+1-2i) p^i, {i, 1, (n+1)/2}];
        g[x + y, x y] // Expand
Out[6]= x^3 + y^3
In[7]:= {u1, u2} =
          Cancel[{(g[x1 + x2, x1 x2] - g[x3 + x4, x1 x2])/v1,
                  (g[x3 + x4, x1 x2] - g[x3 + x4, x3 x4])/v2}]
Out[7]= {x1^2 - x1 x2 + x2^2 + x1 x3 + x2 x3 + x3^2 + x1 x4 + x2 x4 + 2 x3 x4 + x4^2,
         -3 (x3 + x4)}
In[8]:= ω = u1 v1 + u2 v2 // Expand
Out[8]= x1^3 + x2^3 - x3^3 - x4^3
In[9]:= Simplify[Q.Q == ω IdentityMatrix[4]]
Out[9]= True
```

Example: Set $P=d$ 




```
In[10]:=
P = ( 0        0        x1 - x4   x2 - x3   )
    ( 0        0        π_{2,3}   -π_{1,4}  )
    ( π_{1,4}  x2 - x3  0         0         )
    ( π_{2,3}  x4 - x1  0         0         );
In[11]:=
  Simplify[P.P == ω IdentityMatrix[4]]
Out[11]=
  True
```

Theorem: (Kh–Ro) Taking homology and then the graded Euler characteristics, we get the [MOY] relations:



[MOY] := Murakami, Ohtsuki, Yamada, Enseignement Math. 44 (1998)

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$

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