Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y) = e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\text{sder}_n)$ and $\mathcal{U}(\text{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F \in \mathcal{U}(\text{tder}_2); \quad F^{-1}e(x+y)F = e(x)e(y) \quad \Longleftrightarrow \quad F \in \text{Sol}_0$$

$$\Phi = \Phi_F := (F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23} \in \mathcal{U}(\text{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34} \qquad \text{"the pentagon"}$$

$$t = \frac{1}{2}(y,x) \in \text{sder}_2 \text{ satisfies } 4T \quad \text{and} \quad r = (y,0) \in \text{tder}_2 \text{ satisfies } 6T$$

$$R := e(r) \text{ satisfies Yang-Baxter: } R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

$$\text{also } R^{12,3} = R^{13}R^{23} \text{ and } F^{23}R^{1,23}(F^{23})^{-1} = R^{12}R^{13}$$

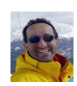
$$\tau(F) := RF^{21}e(-t) \text{ is an involution}, \Phi_{\tau(F)} = (\Phi_F^{321})^{-1}$$

$$\text{Sol}_0^\tau := \{F\colon \tau(F) = F\} \text{ is non-empty; for } F \in \text{Sol}_0^\tau,$$

$$e(t^{13}+t^{23}) = \Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and} \quad e(t^{12}+t^{13}) = (\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev

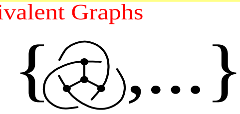

Torossian

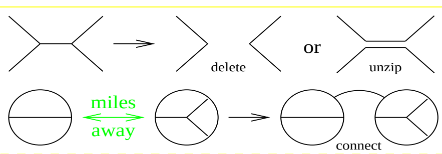
This is just a part of the Alekseev-Torossian work!

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

Knotted Trivalent Graphs

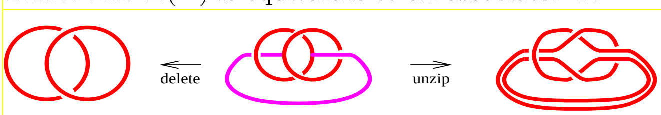
$\mathcal{O}(\triangle) =$




Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.

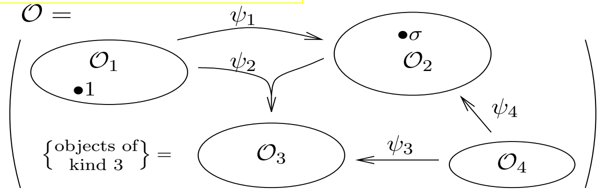
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.

Algebraic Knot Theory



Theorem. $\{\text{ribbon knots}\} \sim \{u\gamma\colon \gamma \in \mathcal{O}(\circ\!\circ),\ d\gamma = \bigcirc\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

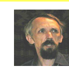
$$I = I_{\mathcal{O}} := \left\{\begin{array}{l}\text{formal differences of ob-}\\ \text{jects "of the same kind"}\end{array}\right\}.$$

Then $I^n := \left\{\begin{array}{l}\text{all outputs of algebraic}\\ \text{expressions at least } n \text{ of}\\ \text{whose inputs are in } I\end{array}\right\}$, and

$$\text{proj}\,\mathcal{O} := \bigoplus_{n\geq 0} I^n/I^{n+1} \quad \left(\begin{array}{l}\text{has same kinds and opera-}\\ \text{tions, but different objects}\\ \text{and axioms}\end{array}\right).$$

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev


Goussarov

Warmup Examples.

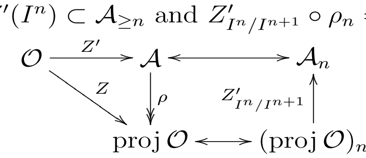
- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x \wedge x = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

proj Q is a graded Lie algebra: set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

An Expansion is $Z\colon \mathcal{O} \to \text{proj}\,\mathcal{O}$ s.t. $Z(I^n) \subset (\text{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}} = Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). **In practice,** it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho\colon \mathcal{A} \to \text{proj}\,\mathcal{O}$. So find $Z'\colon \mathcal{O} \to \mathcal{A}$ with $Z'(I^n) \subset \mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}} \circ \rho_n = Id_{\mathcal{A}_n}$:



Can you make this diagram less confusing?


X-S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.

The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MSRI-0808/