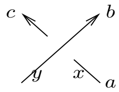
Dror Bar−Natan: Talks: Sandbjerg−0810:

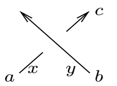
The Penultimate Alexander Invariant

A Definition of the MVA (From [Ar])



$\rightarrow$

	a	b	c
c	−1	$1-x$	y

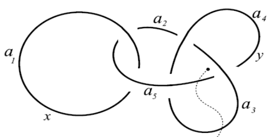


$\rightarrow$

	a	b	c
c	$-y$	$x-1$	1

Joint with Jana Archibald

$$A = \frac{(-)^{i+j}\det(M_i^j)}{w_i(t_i-1)} \prod_k t_k^{\frac{\mathrm{rot}(k)-\mu(k)}{2}}$$

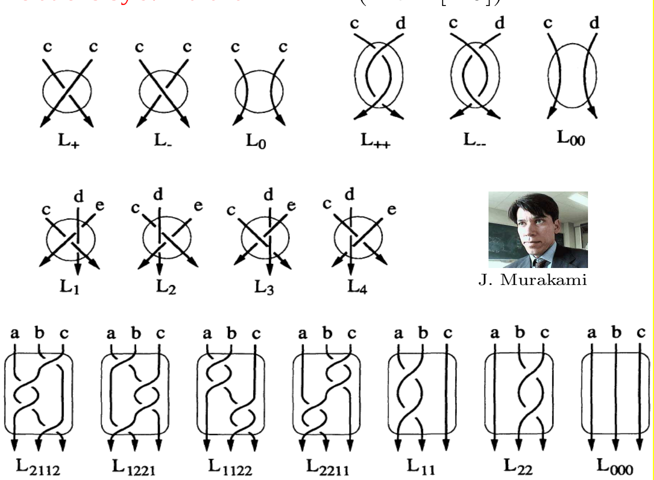


$\rightarrow$

	a_1	a_2	a_3	a_4	a_5
a_1	$y-1$	0	0	0	$1-x$
a_2	$1-y$	x	0	0	-1
a_3	$-y$	0	1	$y-1$	0
a_4	0	0	$-y$	1	$y-1$
a_5	0	0	$y-1$	$-y$	1

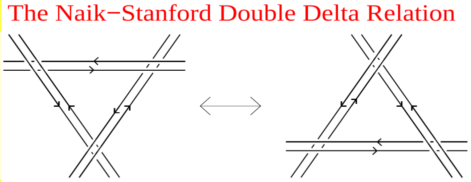
$\rightarrow \quad xy(1-y+y^2)$

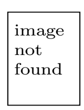
Relations by J. Murakami (From [MJ])



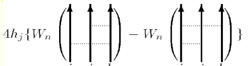
J. Murakami

The Naik−Stanford Double Delta Relation (From [NS])





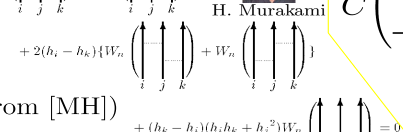
S. Naik T. Stanford

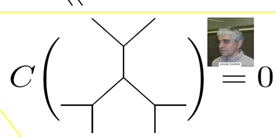


H. Murakami

(From [MH])

A Relation by H. Murakami





A Relation by A. Vaintrob (From [Va])

There's Lots More!

"God created the knots,
all else in topology
is the work of mortals"
Leopold Kronecker (paraphrased)

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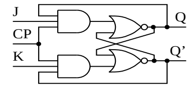
This handout and further links are at
http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg−0810/

Our Goal. Prove all these relations uniformly, at maximal confidence and minimal brain utilization.

$\Rightarrow$ We need an "Alexander Invariant" for arbitrary tangles, easy to define and compute and well-behaved under tangle compositions; better, "virtual tangles".

Circuit Algebras



A J–K Flip Flop

* Have "circuits" with "ends",
* Can be wired arbitrarily.
* May have "relations" − de−Morgan, etc.

Example $V\mathcal{T} = CA\langle \overcrossing, \undercrossing \rangle / R23 = PA\langle \overcrossing, \undercrossing, \virtualcrossing \rangle / R23, VR123, MR3$

Reminders from linear algebra. If X is a (finite) set,

$$\Lambda^k(X) := \langle k\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{\mathrm{top}}(X) := \langle |X|\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle$$

$$\Lambda^{1/2}(X) := \langle (|X|/2)\text{-tuples in } X, \text{ modulo anti-symmetry}\rangle .$$

If $Y \subset X^m$, the "interior multiplication" $i_Y : \Lambda^k(X) \to \Lambda^{k-m}(X)$ is anti-symmetric in Y.

Definition. An "Alexander half density with input strands X^{in} and output strands X^{out}" is an element of

$$\mathrm{AHD}(X^{\mathrm{in}}, X^{\mathrm{out}}) := \Lambda^{\mathrm{top}}(X^{\mathrm{out}}) \otimes \Lambda^{1/2}(X^{\mathrm{in}} \cup X^{\mathrm{out}}).$$

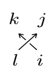
Often we extend the coefficients to some polynomial ring without warning.

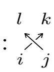
Definition. If $\alpha_i \otimes p_i \in \mathrm{AHD}(X_i^{\mathrm{in}}, X_i^{\mathrm{out}})$ (for $i = 1, 2$), and $G = (X_1^{\mathrm{in}} \cup X_2^{\mathrm{in}}) \cap (X_1^{\mathrm{out}} \cup X_2^{\mathrm{out}})$ is the set of "gluable legs", the "gluing" in $\mathrm{AHD}(X_1^{\mathrm{in}} \cup X_2^{\mathrm{in}} - G, X_1^{\mathrm{out}} \cup X_2^{\mathrm{out}} - G)$ is

$$i_G(\alpha_1 \wedge \alpha_2) \otimes i_G(p_1 \wedge p_2).$$

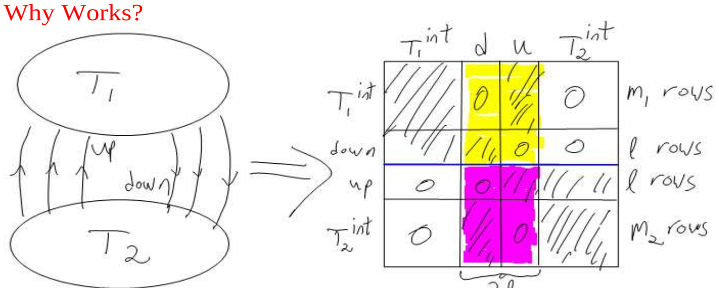
Claim. This makes AHD a circuit algebra.

Definition. The "Penultimate Alexander Invariant" is defined using

pA :  $\mapsto (j \wedge k) \otimes \begin{pmatrix} l \wedge i + (t_i - 1) l \wedge j - t_l l \wedge k \\ +i \wedge j + t_l j \wedge k \end{pmatrix}$

pA :  $\mapsto (k \wedge l) \otimes \begin{pmatrix} t_j i \wedge j - t_j i \wedge l + j \wedge k \\ +(t_i - 1) j \wedge l + k \wedge l \end{pmatrix}$

Why Works?



Every "rook arrangement" in the above picture must have exactly l rooks in the yellow zone and l rooks in the purple zone. So for T_1 we only care about the minors in which exactly l of the $2l$ middle columns are dropped, and the rest is signs...

Weaknesses. Exponential, no understanding of cablings, no obvious "meaning". The ultimate Alexander invariant should address all that...

Challenge. Can you categorify this?

More at http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/