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Algebraic Knot Theory

Colloquium in Mathematics

University of Copenhagen, October 9, 2008

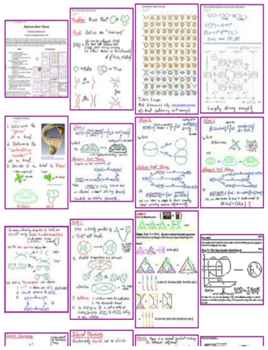
Abstract. The right objects of study in algebraic topology are not spaces, but rather, "spaces and maps between them". In a similar spirit I will argue that the right things to study in knot theory are not knots, but rather, "knotted trivalent graphs", as in the world of knotted trivalent graphs (and the basic operations between them) many interesting properties of honest knots become "definable". Thus I find myself again studying the good old Kontsevich integral - the best example I know of an algebraic knot theory - but my perspective this time is completely different.

Menu

- Some very basic knot theory.
  - The three-colouring invariant.
  - The Rolfsen table.
  - The Kauffman bracket and the Jones polynomial.
- Three things we'd like to understand:
  - The genus of a knot.
  - The crossing number of a knot.
  - Ribbon knots.
- TG-algebras and TG-morphisms.
  - Aside 1: KTG is finitely generated (and presented).
  - Aside 2: And it is related to Drinfel'd associators.
- A word about "definable" sets.
- A very strong TG-morphism exists! (But it is too hard...)
- The envelope of the Alexander polynomial. (Must be doable, but I don't know how).
- Internal quotients. (Likely more than just Lie algebra; may have "moduli" rather than just discrete points).
- Open questions and propaganda.

Also see my talk in Aarhus, June 2007. Much progress was made since, but the introductory talk (this one) remains more or less the same.

Transparencies



The Three-Colouring Invariant

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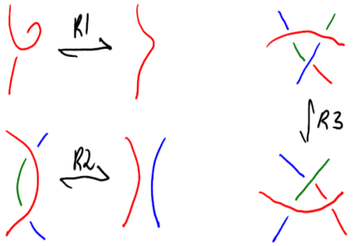
Problem Prove that $\bigcirc \neq$ trefoil.

Proof Define an "invariant"

$I(D) :=$ | D can be coloured RGB so that all colours are used, and every crossing is either mono- or tri-Chromatic | $\in \{\text{True}, \text{False}\}$

$\bigcirc \to$ False $\to$ True



The Rolfsen Table of Knots

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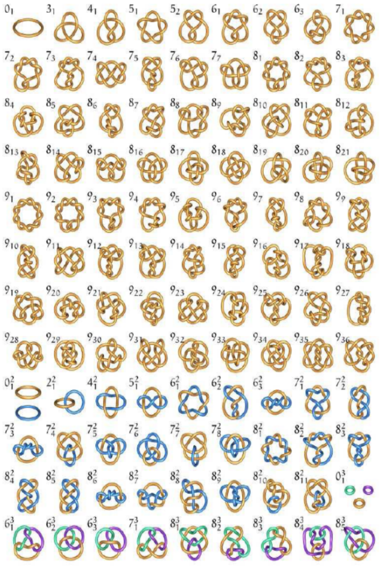

Taken from Rob Scharein's site, http://knotplot.com/zoo/

$\Rightarrow$ Knot Colouring isn't enough.

The Kauffman Bracket and the Jones Polynomial

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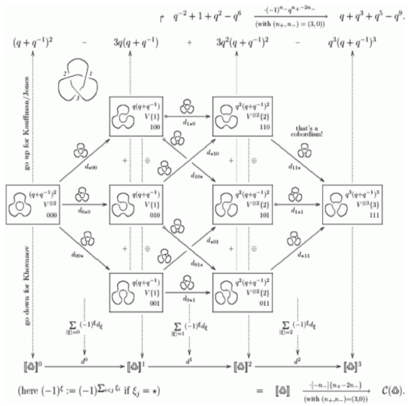
$\langle \times \rangle = \langle \underset{\text{0-smoothing}}{\asymp} \rangle - q \langle \underset{\text{1-smoothing}}{)(} \rangle$

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+-2n_-} \langle L \rangle$

(n_+, n_-) count $(\nwarrow\!\!\!\nearrow, \nearrow\!\!\!\nwarrow)$



Largely strong enough!

Three Basic Problems

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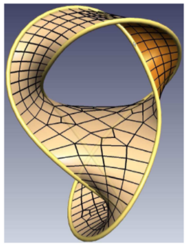
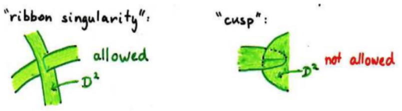
1. Determine the "genus" of a knot.
2. Determine the "unknotting number" of a knot.
3. Decide if a knot is "Ribbon".



Drawn using SeifertView, http://www.win.tue.nl/~vanwijk/seifertview/



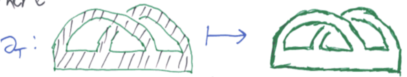
(image by Zsuzsanna Dancso)

Claim 1

$\mathcal{K}(\bigcirc) \supset \left\{\begin{matrix}\text{Knots bounding}\\ \text{a surface of}\\ \text{genus 1}\end{matrix}\right\} = \{\partial_T \gamma : \gamma \in \mathcal{K}(\text{band-graph})\}$

knottings of a band-graph

where



is the "topological boundary" operator.

Algebraic Knot Theory:

Suppose we had invariants Z:

$$\begin{tikzcd}
 & \mathcal{K}(\text{band-graph}) \arrow[r, mapsto, "Z"] \arrow[d, "\partial_T"] & A(\text{band-graph}) \arrow[d, "\partial_A"] \\
\left\{\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right\} = \operatorname{im}\partial_T < & \mathcal{K}(\bigcirc) \arrow[r, mapsto, "Z"] & A(\bigcirc)
\end{tikzcd}$$

Then $Z\left(\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right) \subset \operatorname{im}\partial_A$ and we have an algebraic invariant detecting $\{\text{genus} \geq 1\}$.

Similarly for detecting $\{\text{genus} \geq 7\}$...

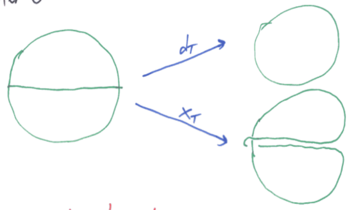
Claim 2

$\mathcal{K}(\bigcirc) \supset \left\{\begin{matrix}\text{Knots of unknotting}\\ \text{number 1}\end{matrix}\right\} = \left\{x_T\gamma : \begin{matrix}\gamma \in \mathcal{K}(\Theta);\\ d_T\gamma = \bigcirc\end{matrix}\right\}$

knottings of Θ

the unknot.

where



Algebraic Knot Theory:

$$\begin{tikzcd}
 & \mathcal{K}(\bigcirc) \arrow[rr, "Z"] & & A(\bigcirc) \\
\mathcal{K}(\Theta) \arrow[ur, "x_T"] \arrow[rr, "Z"] \arrow[dr, "d_T"'] & & A(\Theta) \arrow[ur, "x_A"] \arrow[dr, "d_A"] & \\
 & \mathcal{K}(\bigcirc) \arrow[rr, "Z"] & & A(\bigcirc) \ni \bigcirc
\end{tikzcd}$$

So

$Z\left(\left\{\begin{matrix}\text{unknotting}\\ \text{number 1}\end{matrix}\right\}\right) \subset \left\{x_A\alpha : \begin{matrix}\alpha \in A(\Theta)\\ d_A\alpha = Z(\bigcirc)\end{matrix}\right\}$

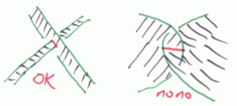
and we stand a chance to learn something about unknotting numbers algebraically.

Claim 3

$\{\text{Ribbon knots}\} \overset{\text{more or less}}{=} \left\{u\gamma : \begin{matrix}\gamma \in \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc)\\ d\gamma = \bigcirc\bigcirc\end{matrix}\right\}$

Where:

Ribbon means



and

Algebraic Knot Theory:

$$\begin{tikzcd}
 & \mathcal{K}(\bigcirc\bigcirc) \arrow[rr, "Z"] & & A(\bigcirc\bigcirc) \ni \bigcirc\bigcirc \\
\mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) \arrow[ur, "d"] \arrow[rr, dash, "Z"] \arrow[dr, "u"'] & & A(\bigcirc\!\!-\!\!\bigcirc) \arrow[ur, "d"] \arrow[dr, "u"] & \\
 & \mathcal{K}(\bigcirc\!\!=\!\!\bigcirc) \arrow[rr, "Z"] & & A(\bigcirc)
\end{tikzcd}$$

So

$Z\left(\left\{\begin{matrix}\text{Ribbon}\\ \text{knots}\end{matrix}\right\}\right) \subset \{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\} \subset A(\bigcirc\!\!-\!\!\bigcirc)$

And we stand a chance to find a counterexample to

$\{\text{Ribbon}\} = \{\text{slice}\}$!

More at http://www.math.toronto.edu/~drorbn/Talks/Copenhagen-081009/