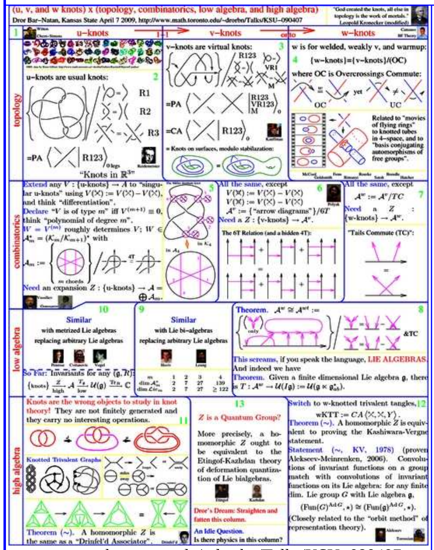
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)



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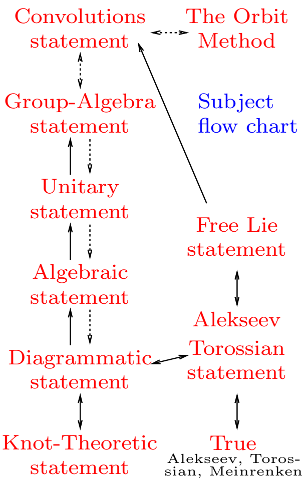
The Bigger Picture...



Convolutions statement

The Orbit Method

Group-Algebra statement

Subject flow chart

Unitary statement

Free Lie statement

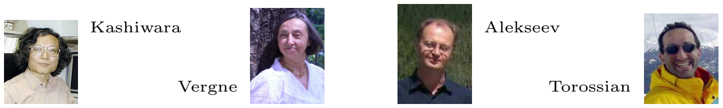
Algebraic statement

Alekseev Torossian statement

Diagrammatic statement

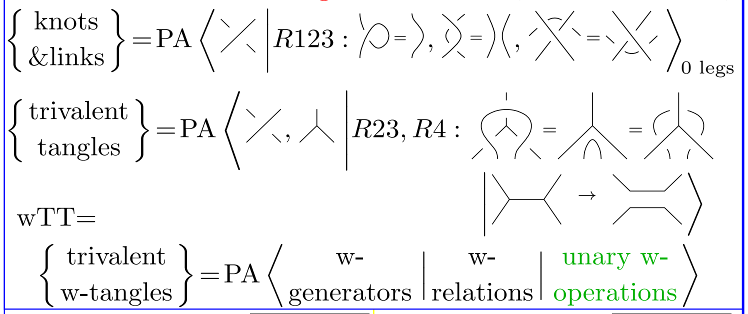
Knot-Theoretic statement

True
Alekseev, Torossian, Meinrenken



What are w-Trivalent Tangles?

(PA :=Planar Algebra)



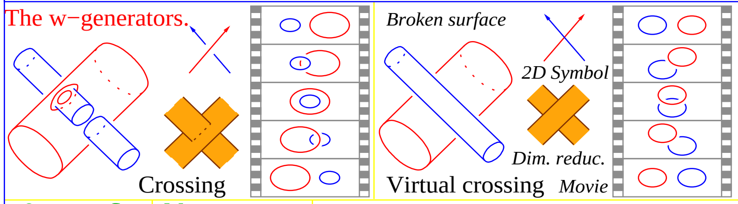
$\left\{\begin{matrix}\text{knots}\\ \text{\&links}\end{matrix}\right\} = PA\left\langle \;\middle|\; R123 : \;\right\rangle_{0\text{ legs}}$

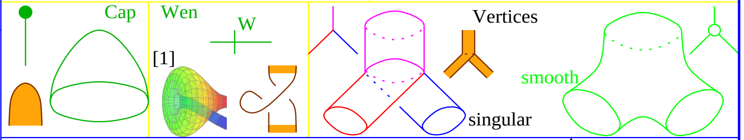
$\left\{\begin{matrix}\text{trivalent}\\ \text{tangles}\end{matrix}\right\} = PA\left\langle \;,\; \middle|\; R23, R4 : \;=\;=\;,\; \to \;\right\rangle$

wTT=

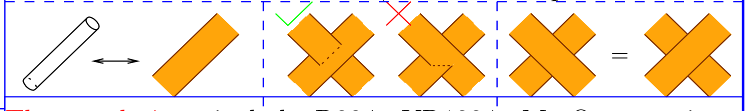
$\left\{\begin{matrix}\text{trivalent}\\ \text{w-tangles}\end{matrix}\right\} = PA\left\langle \text{w-generators} \middle| \text{w-relations} \middle| \text{unary w-operations} \right\rangle$

The w−generators.

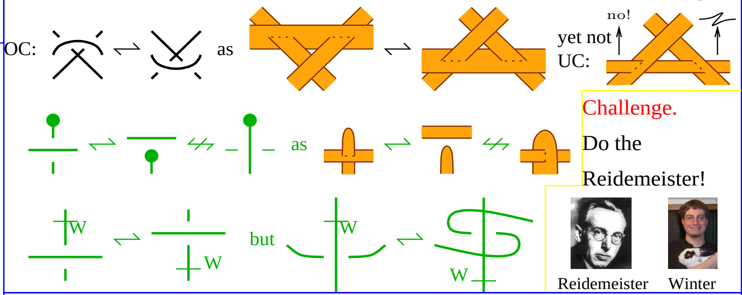




A **Ribbon 2-Knot** is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

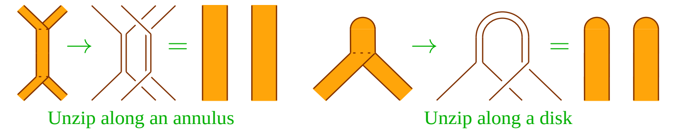


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



Challenge. Do the Reidemeister!

The unary w−operations



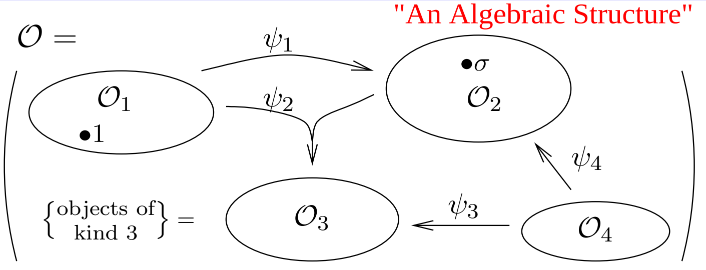
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowright \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \dots \end{array}$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

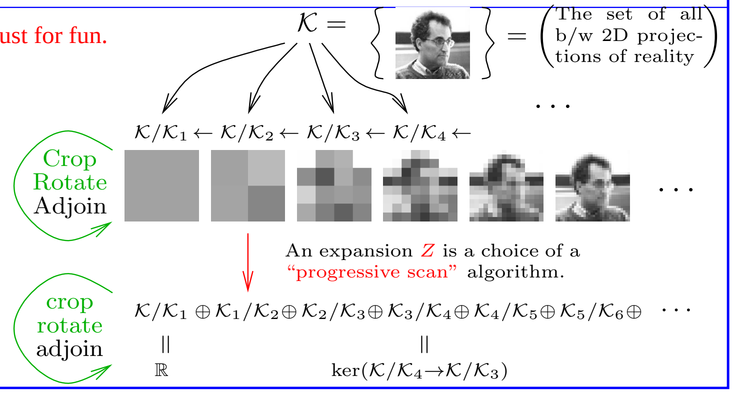
Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{gr}\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

Just for fun.



$\mathcal{K} = \{\;\} = \left(\text{The set of all b/w 2D projections of reality}\right)$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \dots$

An expansion Z is a choice of a "progressive scan" algorithm.

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

[1] http://qlink.queensu.ca/~4lb11/interesting.html

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

29/5/10, 8:42am

Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Bonn-0908/`