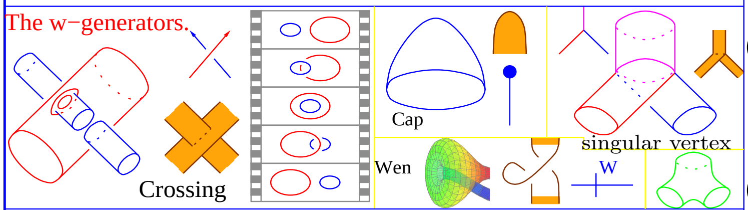
# 2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan

I understand Drinfel'd and Alekseev–Torossian, I don't understand Etingof–Kazhdan yet, and I'm clueless about Kontsevich

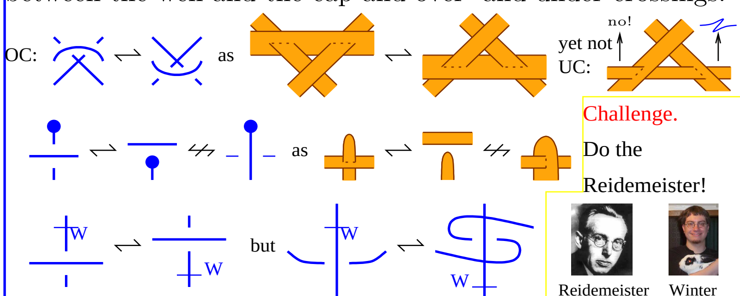
Dror Bar–Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

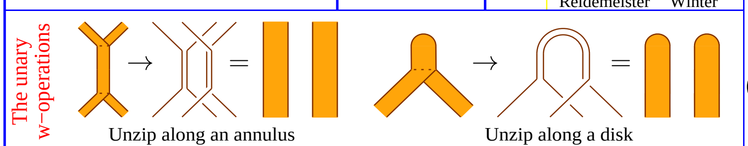
**Trivalent w-Tangles.**

$$\text{wTT} = \text{CA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array} \middle| \begin{array}{c}\text{w-}\\ \text{relations}\end{array} \middle| \begin{array}{c}\text{unary w-}\\ \text{operations}\end{array}\right\rangle$$

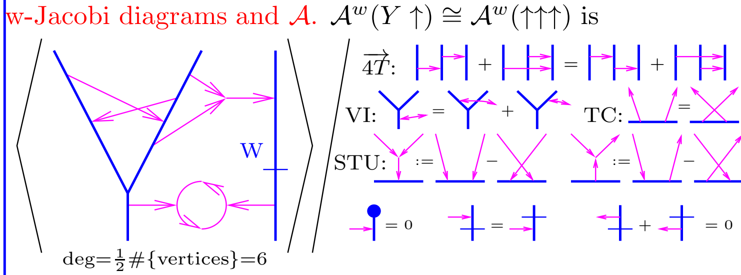



**The w-relations** include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



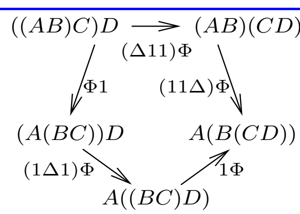

**w-Jacobi diagrams and $\mathcal{A}$.** $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is




**An Associator:**

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

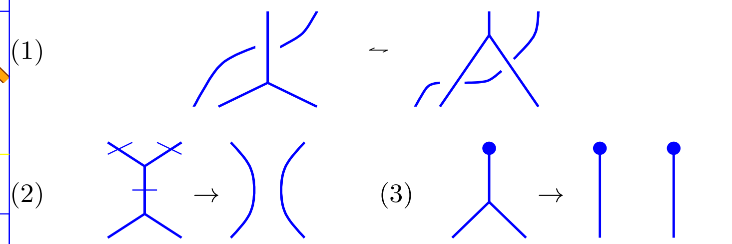
satisfying the "pentagon",



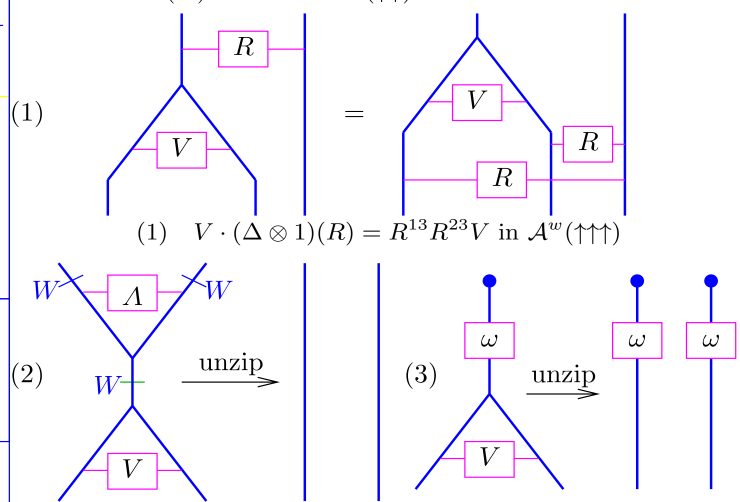

$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

**Knot-Theoretic statement.** There exists a homomorphic expansion $Z$ for trivalent w-tangles. In particular, $Z$ should respect $R4$ and intertwine annulus and disk unzips:




**Diagrammatic statement.** Let $R = \exp \leftrightarrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that




(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

**Alekseev-Torossian statement.** There are elements $F\in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

**Theorem.** The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

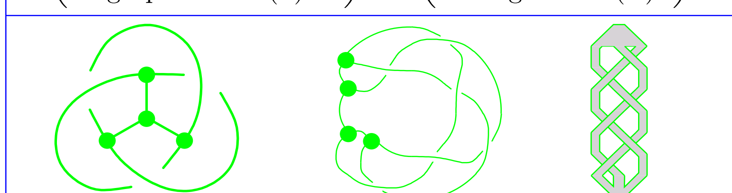
**Proof.** Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b\in\mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$, (2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and (3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$ $\Leftrightarrow c = b(x)+b(y)-b(\log e^x e^y)$.

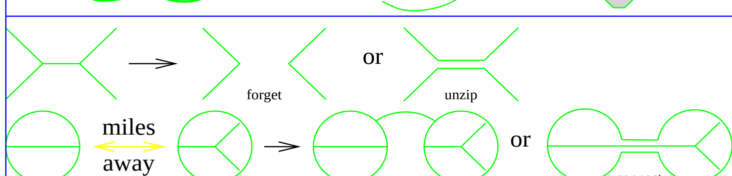
**The Alekseev-Torossian Correspondence.**

$$\{\text{Drinfel'd Associators}\} \leftrightarrows \{\text{Solutions of KV}\}.$$

We need an even bigger algebraic structure!

$$\left(\begin{array}{c}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{array}\right) \xrightarrow{\alpha_e} \left(\begin{array}{c}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{array}\right)$$






Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/