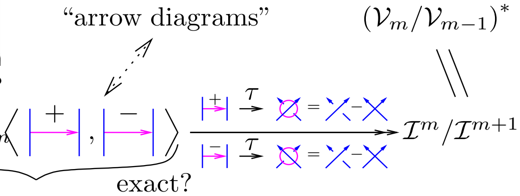
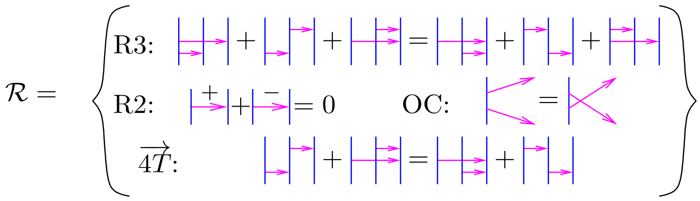
1. proj $\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.

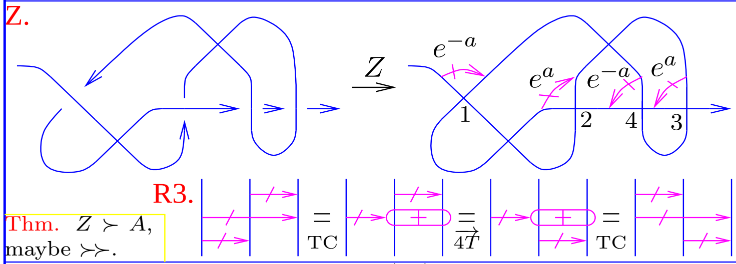
Goussarov-Polyak-Viro



Imperfect Thumb-Rule. Take R3 (say), substitute ⤫ → ⤫ + ⤫, keep the lowest degree terms that don't immediately die:

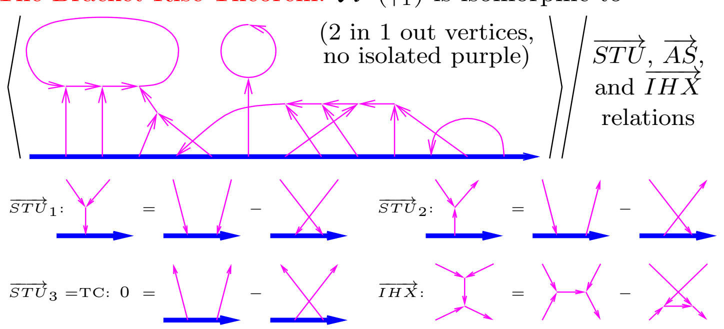


Z.

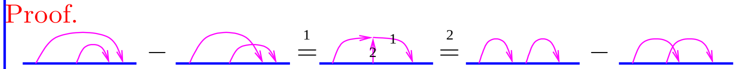


Thm. $Z \succ A$, maybe $\succ\!\succ$.

The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to

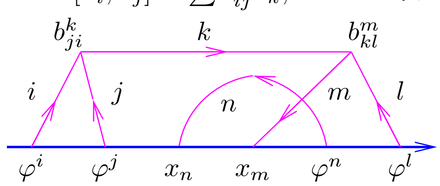


Proof.



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

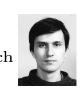
To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



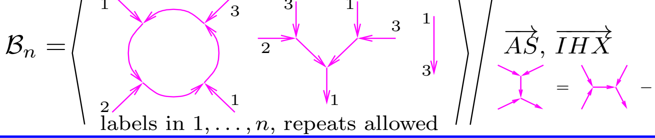
Penrose Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where



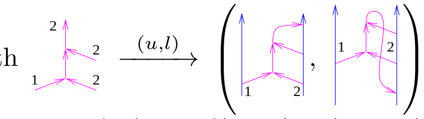
Kontsevich



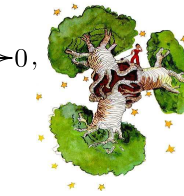
Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \xrightarrow{\iota} \mathcal{PA}^w(\uparrow_n) \underset{l}{\overset{u}{\underset{\pi}{\rightleftarrows}}} \langle\text{trees}\rangle \longrightarrow 0,$$

with



So proj $\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.



trees atop a wheel, and a little prince.

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+ / (x_{i_1} x_{i_2} \cdots x_{i_m} = x_{i_2} \cdots x_{i_m} x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\mathrm{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \mathrm{div}\, D$.

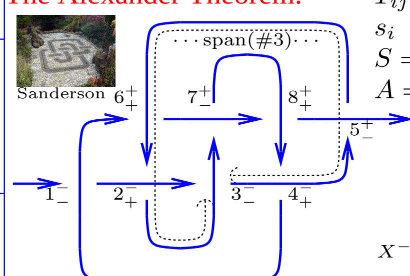
Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\mathrm{div}\, D = \iota^{-1}(u - l)(D)$, and $e^{uD} e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:
- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\, x$: $(x\varphi)(y) := \varphi([x, y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\mathrm{div}\, D$ is $D - D^*$ and $jD = \log(e^D (e^D)^*) = \int_0^1 dt e^{tD}\, \mathrm{div}\, D$.

Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.
Theorem. $\mathfrak{sder}_n = \pi\alpha(\text{proj u-tangles})$, where α is the obvious map proj u-tangles $\to$ proj w-tangles.
Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$$X^{-S} = \mathrm{diag}\left(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}\right).$$

Conjecture. For u-knots, A is the Alexander polynomial.
Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/