18 Conjectures

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http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will state 18 = 3 × 3 × 2 "fundamental" conjectures on finite type invariants of various classes of virtual knots. This done, I will state a few further conjectures about these conjectures and ask a few questions about how these 18 conjectures may or may not interact.

Following *"Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots"*, by B-N, Halacheva, Leung, and Roukema, http://www.math.toronto.edu/~drorbn/papers/v-Dims/.

LRHB by Chu

Theorem. For u-knots, $\dim \mathcal{V}_n/\mathcal{V}_{n-1} = \dim \mathcal{W}_n$ for all n.

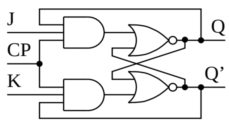
Proof. This is the Kontsevich integral, or the "Fundamental Theorem of Finite Type Invariants". The known proofs use QFT-inspired differential geometry or associators and some homological computations.

Two tables. The following tables show $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$ and $\dim \mathcal{W}_n$ for $n = 1, \dots, 5$ for 18 classes of v-knots:

relations\skeleton		round (◯)	long (⟶)	flat (✕ = ✕)
standard	mod R1	0, 0, 1, 4, 17 ●	0, 2, 7, 42, 246 ●	0, 0, 1, 6, 34 ●
R2b R2c R3b	no R1	1, 1, 2, 7, 29	2, 5, 15, 67, 365	1, 1, 2, 8, 42
braid-like	mod R1	0, 0, 1, 4, 17 ●	0, 2, 7, 42, 246 ●	0, 0, 1, 6, 34 ●
R2b R3b	no R1	1, 2, 5, 19, 77	2, 7, 27, 139, 813	1, 2, 6, 24, 120
R2 only	mod R1	0, 0, 4, 44, 648	0, 2, 28, 420, 7808	0, 0, 2, 18, 174
R2b R2c	no R1	1, 3, 16, 160, 2248	2, 10, 96, 1332, 23880	1, 2, 9, 63, 570

18 Conjectures. These 18 coincidences persist.

Circuit Algebras

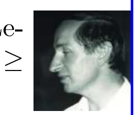


A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

Comments. 0, 0, 1, 4, 17 and 0, 2, 7, 42, 246. These are the "standard" virtual knots.

2, 7, 27, 139, 813. These best match Lie bi-algebra. Leung computed the bi-algebra dimensions to be ≥ 2, 7, 27, 128.



Vogel

●●●. We only half-understand these equalities.

1, 2, 6, 24, 120. Yes, we noticed. Karene Chu is proving all about this, including the classification of flat knots.

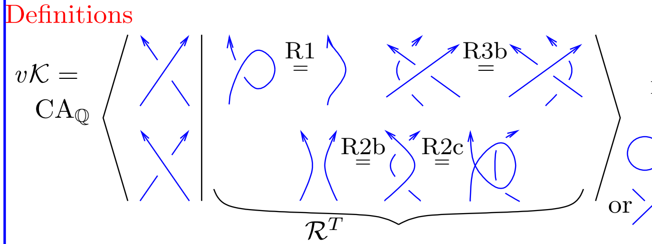
1, 1, 2, 8, 42, 258, 1824, 14664, . . ., which is probably http://www.research.att.com/~njas/sequences/A013999.

What about w? See other side. What about flat and round? Likely **fails!**

What about v-braids? I don't know.

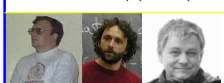
Definitions

Kauffman

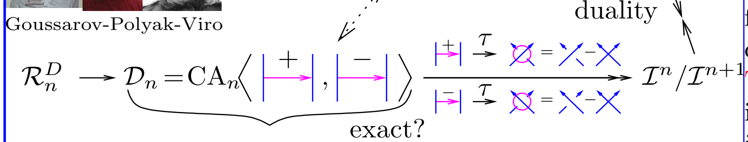


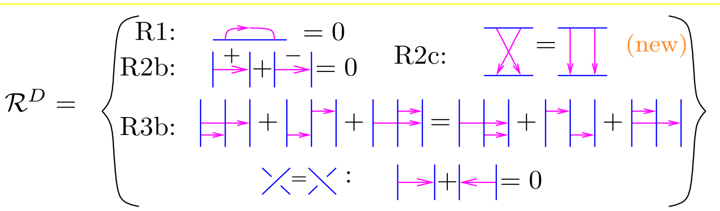
$\mathcal{I} = I\langle$ ✕ $=$ ✕ $-$ ✕ $\rangle$

$\mathcal{V}_n = (v\mathcal{K}/\mathcal{I}^{n+1})^*$ is one thing we measure. . .



Goussarov-Polyak-Viro

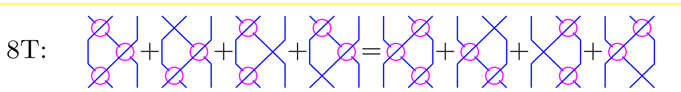




$\mathcal{W}_n = (\mathcal{D}_n/\mathcal{R}^D_n)^* = (\mathcal{A}_n)^*$ is the other thing we measure. . .

The Polyak Technique

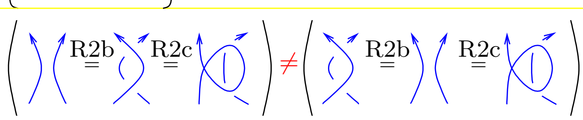
$v\mathcal{K} = \mathrm{CA}_\mathbb{Q}\langle$ ✕ $\rangle / \mathcal{R}^\circ = \{8T, \text{etc.}\}$ fails in the u case



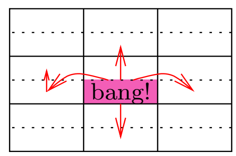
This is a computable space! $\left\{ \mathrm{CA}^{\le n}_\mathbb{Q}\langle ✕ \rangle / \mathcal{R}^{\circ \le n} = v\mathcal{K}/\mathcal{I}^{n+1} \right.$

$$\mathcal{R}^D_n \longleftrightarrow \left\{\begin{array}{c}\text{degree } n\\ \text{"bottoms" of}\\ \text{relations in } \mathcal{R}^\circ\end{array}\right\} \longrightarrow \mathcal{D}_n \xrightarrow{\tau} \mathcal{I}^n/\mathcal{I}^{n+1}$$

Warning!



The True Count



One bang! and five compatible transfer principles.

Bang. Recall the surjection $\bar\tau : \mathcal{A}_n = \mathcal{D}_n/\mathcal{R}^D_n \to \mathcal{I}^n/\mathcal{I}^{n+1}$. A filtered map $Z : v\mathcal{K} \to \mathcal{A} = \bigoplus \mathcal{A}_n$ such that $(\mathrm{gr} Z) \circ \bar\tau = I$ is called a universal finite type invariant, or an "expansion".

Theorem. Such Z exist iff $\bar\tau : \mathcal{D}_n/\mathcal{R}^D_n \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism for every class and every n, and iff the 18 conjectures hold true.

The Big Bang. Can you find a "homomorphic expansion" Z — an expansion that is also a morphism of circuit algebras? Perhaps one that would also intertwine other operations, such as strand doubling? Or one that would extend to v-knotted trivalent graphs?

- Using generators/relations, finding Z is an exercise in solving equations in graded spaces.
- In the u case, these are the Drinfel'd pentagon and hexagon equations.
- In the w case, these are the Kashiwara-Vergne-Alekseev-Torossian equations. Composed with $\mathcal{T}_\mathfrak{g} : \mathcal{A} \to \mathcal{U}$, you get that the convolution algebra of invariant functions on a Lie group is isomorphic to the convolution algebra of invariant functions on its Lie algebra.
- In the v case there are strong indications that you'd get the equations defining a quantized universal enveloping algebra and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. That's why I'm here!

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/