From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

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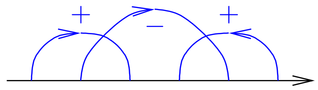
http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

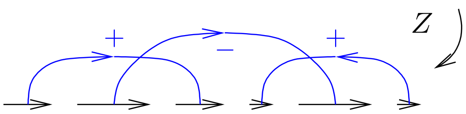
Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing the "Euler technique" and some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

The 2D Lie Algebra. Let $\mathfrak{g} = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$ with $\phi_i(x^j) = \delta_i^j$, let $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ so $[\phi_i, \phi_j] = [\phi_1, x^i] = 0$ while $[x^1, \phi_2] = -\phi_2$ and $[x^2, \phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$. Let $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a,b]$, degree-completed with respect to $\deg \phi_i = 1$ and $\deg x^i = 0$ (so $\mathcal{U} \equiv$ (power series is 4 variables)). Let $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$.

The Invariant. Define Z : {long knots} $\to \mathcal{U}$ by mapping every $\pm$-crossing to $R^{\pm 1}$:



Alexander

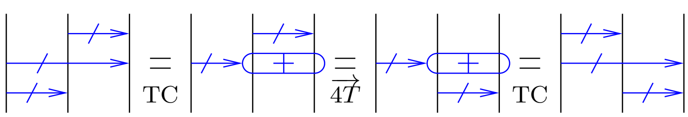


$$\cdots + \frac{1}{2!}\frac{(-1)^3}{3!}\frac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1) + \cdots$$

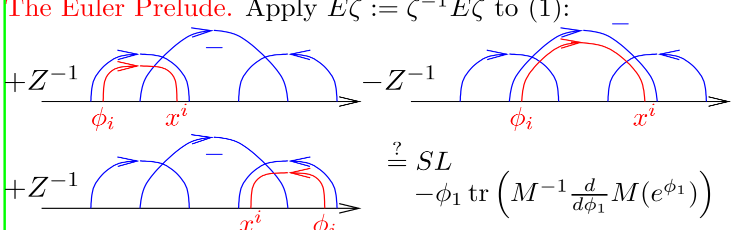
Near Theorem. Z is invariant, and it is essentially the Alexander polynomial; with $N = \exp(\overrightarrow{l}\,\phi_i x^i + \overleftarrow{l}\,x^i\phi_i) =: \exp(SL)$,

$$Z(K) = N \cdot \left(A(K)(e^{\phi_1})\right)^{-1} \qquad (1)$$

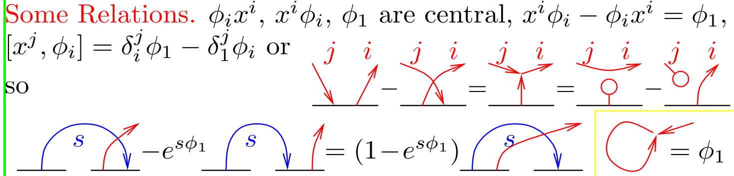
Invariance. "The identity is an invariant tensor":



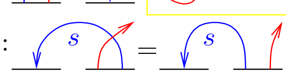
The Euler Prelude. Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$ to (1):



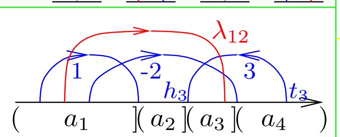
Some Relations. $\phi_i x^i$, $x^i\phi_i$, ϕ_1 are central, $x^i\phi_i - \phi_i x^i = \phi_1$, $[x^j, \phi_i] = \delta_i^j\phi_1 - \delta_1^j\phi_i$ or so



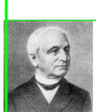
and the famed "tails commute" (TC):



Near Proof. Let $\lambda_{\alpha j}$ be a red arrow with tail at a_α and head just left of h_j. Let $\Lambda = (\lambda_{\alpha j})$. Then roughly $R\Lambda = \phi_1 I$ so roughly, $\Lambda = R^{-1}\phi_1$. The rest is book-keeping that I haven't finished yet, yet with which my computer agrees fully.



I don't understand the Alexander polynomial!



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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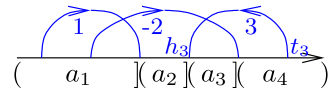
An Alexander Reminder. Number the arrows $1, \ldots, n$, let t_j, h_j be the tail and head of arrow j, and let $s_j \in \pm 1$ be its sign. Cut the skeleton into arcs a_α by arrow heads, and let $\alpha(p)$ be "the arc of point p". Let $R \in M_{n\times(n+1)}$ be the matrix whose j'th row has -1 in column $\alpha(h_j)$ and $1 - X^{s_j}$ in column $\alpha(t_j)$ and X^{s_j} in column $\alpha(h_j)+1$, and let M be R with a column removed. Then $A(X) = \det(M)$.



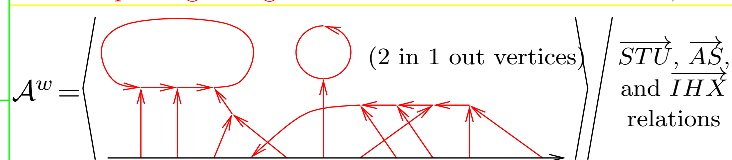
$$\begin{pmatrix} 1-X+(-1) & X & 0 & 0 \\ 1-X^{-1} & 0 & -1 & X^{-1} \\ 0 & -1 & X & 1-X \end{pmatrix}$$

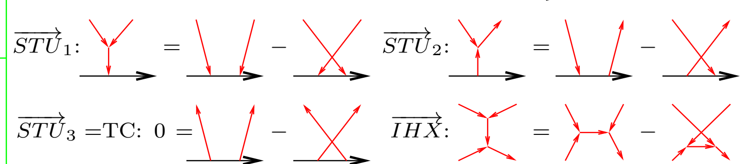
An Euler Interlude. If you know brackets, how do you test exponentials? When's $e^Ae^B = e^Ce^D$?

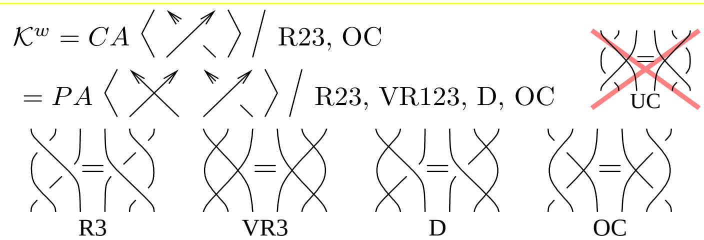
Bad Idea. Take log and use BCH. You'll want to cry.

Clever Idea. Let E be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\![\phi]\!]$, $Ef = \phi\partial_\phi f$, so $Ee^\phi = \phi e^\phi$). Apply $\tilde{E}\zeta := \zeta^{-1}E\zeta$: $\tilde{E}(e^Ae^B) = e^{-B}e^{-A}\left(e^AAe^B + e^Ae^BB\right) = e^{-B}Ae^B + B = e^{-\operatorname{ad}B}(A) + B$.

"Uninterpreting" Diagrams. Make $Z^w : \mathcal{K}^w \to \mathcal{A}^w \to \mathcal{U}$, with

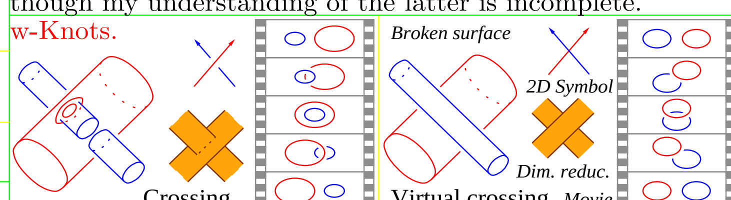


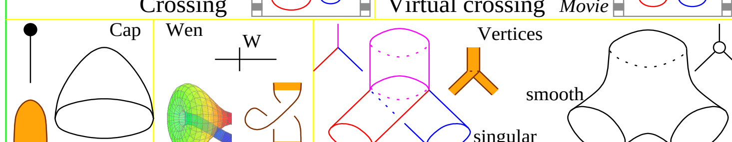




Z^w is a UFTI on w-knots! It extends to links and tangles, is well behaved under compositions and cables, and remains computable for tangles. It contains Burau, Gassner, and Cimasoni-Turaev in natural ways, and it contains the MVA though my understanding of the latter is incomplete.

w-Knots.





There's 1D in 4D, non-trivial given 2D, and there are ops...

Dream. Z^w extends to virtual knots as $Z^v : \mathcal{K}^v \to \mathcal{A}^v$, with good composition and cabling properties and plenty of computable quotients, more then there are quantum groups and representations thereof. I don't understand quantum groups!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/