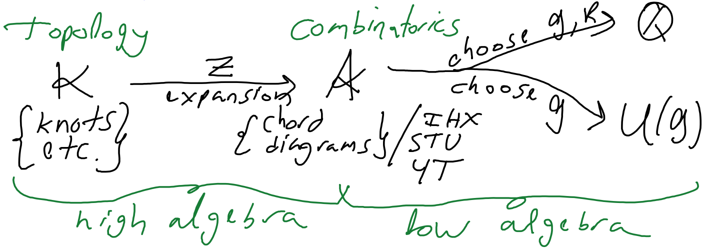
Lecture 4 Handout

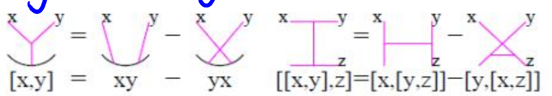
Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.



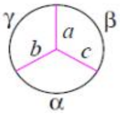
Very low algebra.



More precisely, let $\mathfrak g = \langle X_a\rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha\rangle$ be a representation. Set

$$f_{abc} := \langle [a,b],c\rangle \qquad X_a v_\beta = \sum_{\cancel{\beta}\gamma} r^\beta_{a\gamma} v_\gamma$$

and then



$$W_{\mathfrak g,R}: \longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Excrsice. Find a fast method to find $W_{\mathfrak g,R}(D)$ when $\mathfrak g = gl_n$, $R = \mathbb R^n$? Is it related to the Conway polynomial?

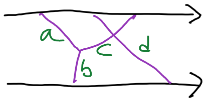
Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal U(\mathfrak g) = \langle \text{words in } \mathfrak g\rangle / [x,y] = xy - yx$

- * Every rep of $\mathfrak g$ extends to $\mathcal U(\mathfrak g)$.
- * $\exists\, \Delta : \mathcal U(\mathfrak g) \to \mathcal U(\mathfrak g)^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak g = \langle x,y\rangle / [x,y] = x$, determine $\mathcal U(\mathfrak g)$. Guess a generalization.

Low algebra. $\mathcal A(\uparrow\uparrow) \to \mathcal U(\mathfrak g)^{\otimes 2}$ via

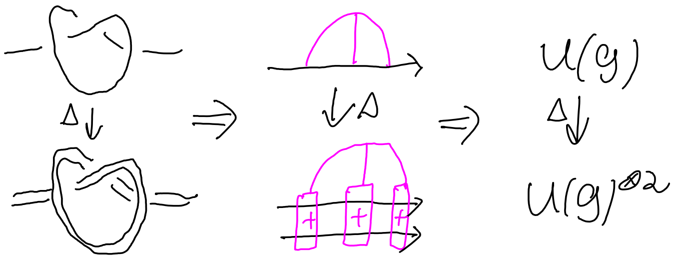


$$\longrightarrow \sum_{a-d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

& likewise, $\mathcal A(\uparrow_n) \to \mathcal U(\mathfrak g)^{\otimes n} \Rightarrow$

$\mathcal A(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

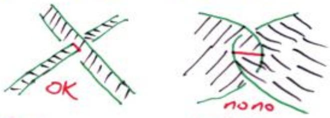


A "Homomorphic Expansion" $Z : \mathcal K \to \mathcal A$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal K$ is finitely presented, finding Z is **High Algebra.**

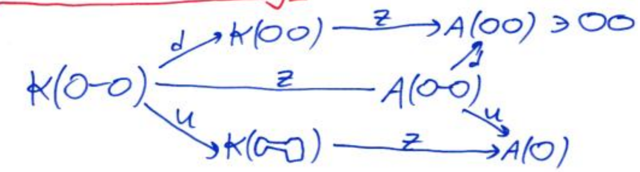
$$\{\text{Ribbon knots}\} \underset{\substack{\uparrow \\ \text{more or less}}}{=} \left\{ u\gamma : \begin{matrix} \gamma \in \mathcal K(\bigcirc\!\!-\!\!\bigcirc) \\ d\gamma = \bigcirc\bigcirc \end{matrix} \right\}$$

where Ribbon means



and

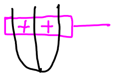
Algebraic knot theory:



So

$$Z\left(\left\{\begin{matrix}\text{Ribbon} \\ \text{knots}\end{matrix}\right\}\right) \subset \{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\} \subset \mathcal A(\bigcirc\!\!-\!\!\bigcirc)$$

VI



$= 0$, follows from

An Associator:

Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal U(\mathfrak g)^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{tikzcd}
((AB)C)D \arrow[rr, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"'] & & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\
(A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & & A(B(CD)) \\
& A((BC)D) \arrow[ur, "1\Phi"'] &
\end{tikzcd}$$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

V.G. Drinfel'd

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107/