

Lecture 3 Handout

The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, <http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107>

Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.

$$C: \left\{ \begin{array}{l} \text{Knots} \\ \text{with } \chi_1 = \chi_2 \end{array} \right\} \rightarrow \mathbb{Z}[z]$$

Example. The Conway polynomial is given by

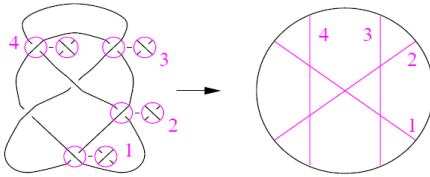
$$C(\text{crossing}) - C(\text{smoothing}) = z C(\text{other crossing})$$

$$\text{and } C(\text{link with } k \text{ crossings}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$$

Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.



Definition. Any $V: \{\text{knots}\} \rightarrow \text{Abelian Group } A$ can be extended to "knots w/ double points" using $V(\text{crossing}) = V(\text{smoothing}) - V(\text{other smoothing})$. (Think "differentiation")



Definition. V is of type m if always $V(\text{link with } m+1 \text{ crossings}) = 0$ (think "polynomial")

$$V(\text{link with } m+1 \text{ crossings}) = 0$$

Conjecture. Finite type invariants separate knots.

Theorem. If $C(k) = \sum_{m=0}^{\infty} V_m(k) z^m$ then V_m is of type m .

Proof. $C(\text{crossing}) = C(\text{smoothing}) - C(\text{other smoothing}) = z C(\text{other crossing}) \square$

Let V be of type m ; then $V^{(m)}$ is constant:

$$V(\text{link with } m \text{ crossings}) = V(\text{link with } m-1 \text{ crossings})$$

So $W_V := V^{(m)} = V|_{\text{m-singular knots}}$ is really a function on m -chord diagrams: $W_V: \{\text{m-chord diagrams}\} \rightarrow A$

Claim. W_V satisfies the 4T relation:

$W_V(\text{diagram 1}) - W_V(\text{diagram 2}) - W_V(\text{diagram 3}) + W_V(\text{diagram 4}) = 0$

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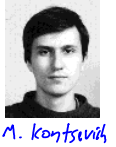
Proof. $V(\text{diagram 1}) = V(\text{diagram 2}) - V(\text{diagram 3}) + V(\text{diagram 4}) \square$

Exercise for Lecture 2. Use $\int_{\mathbb{R}^n} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-||x||^2/2} dx$ in two different ways and hence to deduce the volume of S^{n-1} , the $(n-1)$ -dimensional sphere.

Exercise. 1. Determine the "weight system" W_m of the m -th coefficient of the Conway polynomial and verify that it satisfies 4T. 2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

Theorem. (The Fundamental Theorem)

Every "weight system", i.e. every linear functional W on $A := \{\text{chord diagrams}\} / 4T$ is the m th derivative of a type m invariant: $\forall W \exists V$ s.t. $W = W_V$



m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim A_m^r$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim A_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim P_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Theorem. $A^{\text{today}} \cong A^{\text{Monday}}$

Proof

$$\text{diagram 1} - \text{diagram 2} = \text{diagram 3} = \text{diagram 4} - \text{diagram 5} \square$$

Proposition. The fundamental theorem holds iff there exists an expansion:

$Z: K \rightarrow \hat{A}$ s.t. if K is m -singular, then $Z(K) = D_k + \text{higher degrees}$.

Proof.

$$K \xrightarrow{Z} \hat{A} \xrightarrow{W} \mathbb{Q} \square$$

Also see my old paper, "On the Vassiliev knot invariants" (google will find...)

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KnotTheory
Loading KnotTheory' version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.

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