More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \mapsto A/\sqrt{\hbar}$, and setting $\hbar = \frac{1}{k}$:

$$Z(\gamma) = \int_{A \in \mathcal{A}'(\mathbb{R}^3, \mathfrak{g})} \mathcal{D}A \; tr_R\, hol_\gamma(A)\; e^{\frac{i}{4\pi}\int_{\mathbb{R}^3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)}$$

$CS(A)$

where $tr_R\, hol_\gamma(A) = tr_R\left(1 + \hbar\int ds\, A(\dot\gamma(s)) + \hbar^2 \int_{s_1<s_2} A(\dot\gamma(s_1))A(\dot\gamma(s_2)) + \cdots\right)$

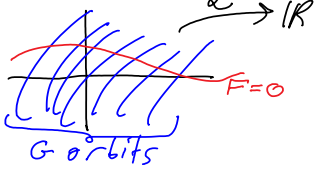
Trouble? "d" is not invertible?

Gauge Invariance: $CS(A)$ is invariant under $A \mapsto A + \delta A$, $\delta A = -(dC + \hbar[A, C])$, $C \in \mathcal{A}^0(\mathbb{R}, \mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional group G w/ Lie algebra $\mathfrak{g} = \langle g_a \rangle$, and suppose $F: \mathbb{R}^n \to \mathbb{R}^k$ is such that $F=0$ is a section of the G-action:

$G \to \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$



Then

$$\int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \delta(F(x)) \cdot \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^l} d\phi\; e^{i(\mathcal{L} + F(x)\cdot\phi)} \det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

perturbation theory for determinants

$$\det(J_0 + \hbar J_1(x)) = \det(J_0)\sum_m \hbar^m\, Tr\left(\Lambda^m J_0^{-1}\right)\cdot\left(\Lambda^m J_1(x)\right)$$

Berezin Fermionic Anti-commuting Variables: $\int d^k\bar c\, d^k c\; e^{i\bar c_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar c \int d^k c\; e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar c_a\left(\frac{\partial F^a}{\partial g_b}\right)c^b}_{\text{"ghosts"}}$$

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

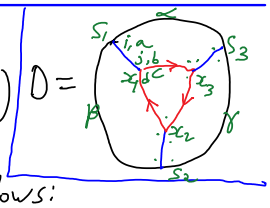
$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \phi\,\partial_i A^i + \bar c\,\partial_i(\partial^i + ad\,A^i)c\right)$$

So we have

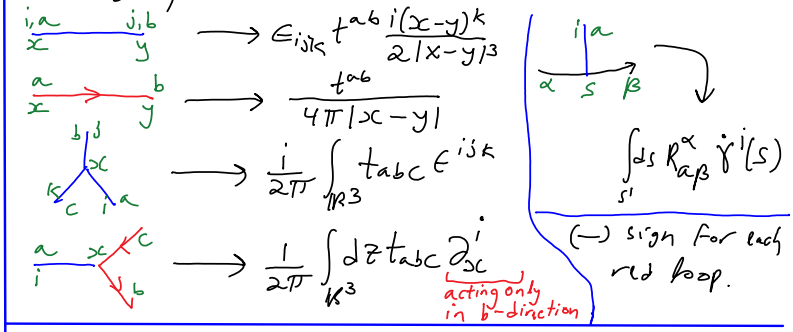
* A bosonic quadratic term involving $\binom{A}{\phi}$.
* A fermionic quadratic term involving $\bar c, c$.
* A cubic interaction of 3 A's.
* A cubic $A\bar c c$ vertex.
* Funny A and γ "holonomy" vertices along γ.

After much crunching:

$$Z(\gamma) = \sum_{m=0}^{\infty} \hbar^m \sum_{\substack{\text{Feynman}\\ \text{diags } D}} \oint \mathcal{E}(D)\, D$$



where $\mathcal{E}(D)$ is constructed as follows:



By a bit of a miracle, this boils down to a configuration space integral, which in itself can be reduced to a pre-image count.

... But I run out of steam for tonight...

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

Banks like knots.