


Example.



(goes back to [Koh])

$$K = \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle \quad I = \left\langle \begin{array}{c} \times \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle$$

$(K/I^{p+1})^* = (\text{invariants of type } p) =: \mathcal{V}_p$


$$(I^p/I^{p+1})^* = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \text{HH} \rangle$$

$$\ker \bar{\mu}_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4T \text{ relations} \rangle$$

$$A = q(K) = \left(\text{horizontal chord diagrams mod } 4T \right) = \langle \text{HH} \rangle / 4T$$

Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \leq i \neq j \leq n \rangle / \begin{array}{l} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{array}$$



L. Kauffman [Kau, KL]

of “pure virtual braids” (“braids when you look”, “blunder braids”):

$$\sigma_{24} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \end{array} \quad R3: \begin{array}{c} \uparrow^k \quad \uparrow^j \quad \uparrow^i \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow^i \quad \uparrow^j \quad \uparrow^k \end{array} = \begin{array}{c} \uparrow^k \quad \uparrow^j \quad \uparrow^i \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow^i \quad \uparrow^j \quad \uparrow^k \end{array}$$

The Main Theorem [Lee]. PvB_n is quadratic.

$A_n = q(PvB_n)$.



[GPV] Goussarov-Polyak-Viro

$$I = \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle \quad \text{with } \bowtie = \tilde{\sigma}_{ij} = \sigma_{ij} - 1 = \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array},$$

the “semi-virtual crossing”.

$$V = I/I^2 = \left\langle \begin{array}{c} \text{v-braids} \\ \text{with one } \bowtie \end{array} \right\rangle / (\bowtie = \times)$$

$$= \langle a_{ij} \rangle_{1 \leq i \neq j \leq n} \quad a_{24} = \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \end{array}$$

$$A_n = TV / \langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}], c_{kl}^{ij} = [a_{ij}, a_{kl}] \rangle,$$

$$y_{ijk} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

$$I^p: \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

James Gillespie’s Sightline #2 (1984) is a syzygy, and (arguably) Toronto’s largest sculpture. Find it next to University of Toronto’s Hart House.



$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C_{kl}^{ij} and

$$Y_{ijk} := \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

$$- \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

Syzygy Completeness, for PvB_n , means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^p \xrightarrow{\pi} V^{\otimes p}$$

$$\{\tilde{\sigma}_{12} : Y_{345} : \tilde{\sigma}_{67} : \dots\} \longrightarrow \{\tilde{\sigma}_{12} : Y_{345} : \tilde{\sigma}_{67} : \dots\} \longrightarrow \{a_{12}y_{345}a_{67} : \dots\}$$

Is every relation between the y_{ijk} ’s and the c_{kl}^{ij} ’s also a relation between the Y_{ijk} ’s and the C_{kl}^{ij} ’s?

The Group PvB_n

Generators: $\sigma_{ij} \rightarrow \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \end{array}$

Relations:

$$C_{kl}^{ij}: \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \rightarrow \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

$$Y_{ijk}: \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \rightarrow \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

A Syzygy:

Theorem S. Let D be the free associative algebra generated by symbols a_{ij} , y_{ijk} and c_{kl}^{ij} , where $1 \leq i, j, k, l \leq n$ are distinct integers. Let D_0 be the part of D with only a_{ij} symbols and let D_1 be the span of the monomials in D having only a_{ij} symbols, with exactly one exception that may be either a y_{ijk} or a c_{kl}^{ij} . Let $\partial : D_1 \rightarrow D_0$ be the map defined by

$$y_{ijk} \mapsto [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],$$

$$c_{kl}^{ij} \mapsto [a_{ij}, a_{kl}].$$

Then $\ker \partial$ is generated by a family of elements readable from the picture above and by a few similar but lesser families.