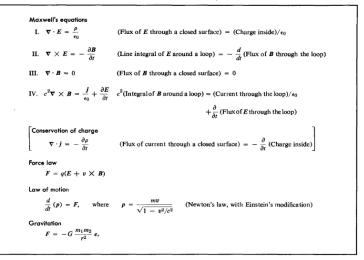
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A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on \mathbb{R}^n , every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.
- The Hodge star operator \star which satisfies $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when F = ma.

Table 18-1 Classical Physics



The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The Vector Field is a compactly supported 1-form A on \mathbb{R}^4 which extremizes the action

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} ||dA||^2 dt dx dy dz + J \wedge A$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless dJ = 0, and that we might as well (think Poincaré's Lemma!) change variables to F := dA. We thus get

$$dJ = 0$$
 $dF = 0$ $d \star F = J$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dx dy dz - j_x dy dz dt - j_y dz dx dt - j_z dx dy dt$, we find:

| $dJ = 0 \Longrightarrow$ | $\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$ | "conservation of charge" |
|-----------------------------|---|---------------------------------|
| $dF = 0 \Longrightarrow$ | $\operatorname{div} B = 0$ | "no magnetic monopoles" |
| | $\operatorname{curl} E = -\frac{\partial B}{\partial t}$ | that's how generators work! |
| $d * F = J \Longrightarrow$ | $\operatorname{div} E = -\rho$ | "electrostatics" |
| | $\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$ | that's how electromagnets work! |

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers. **Exercise.** With $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc \int_{e_1}^{e^2} (ds + eA)$ to derive Feynman's "law of motion" and "force law".

November 30, 2011; http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2011-11#OtherFiles

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