With $\omega\epsilon\beta :=$ http://www.math.toronto.edu/drorbn/Talks/Hamburg-1208

A Quick Introduction to Khovanov Homology

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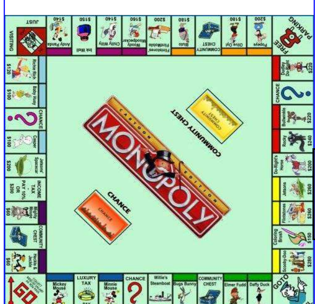
Abstract. I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff — the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of sl_2 and other Lie algebras.

Why Bother?



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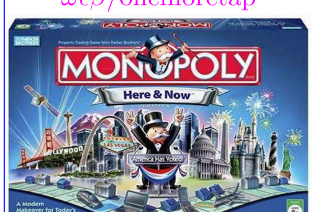

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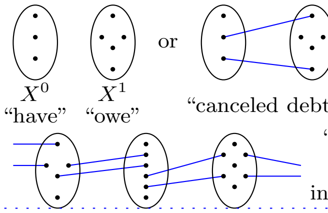
What is Categorification=Concretization=de-abstraction? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things…

…categorification is choosing which 3 it is!

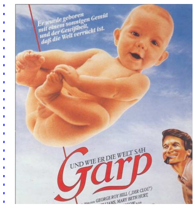
$\mathbb{N}$. Natural numbers $\mapsto$ finite sets, equalities $\mapsto$ bijections, inequalities $\mapsto$ injections and surjections:

$$\binom{2n}{n} = \sum\binom{n}{k}^2 \mapsto \binom{X\times\{1,2\}}{|X|} \leftrightarrow \bigcup \binom{X}{k}\times\binom{X}{k}$$

$\mathbb{Z}$. Negative numbers:



The Philosophy Corner

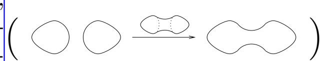


Weaker Categorification. Do the same in the category of vector spaces: "3" becomes V s.t. $\dim V = 3$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$ and $\chi(V^\bullet) := \sum(-1)^r\dim V^r = 3 = \sum(-1)^r\dim H^r$. Equalities become homotopies between complexes.

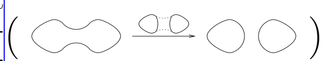
Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;
$V=\operatorname{span}\langle v_+, v_-\rangle$; $\deg v_\pm = \pm 1$; $q\dim V = q+q^{-1}$;

$$K(\bigcirc^k) = V^{\otimes k};\quad K(\times) = \text{Flatten}\Big(0\to \underbrace{K(\asymp)\{1\}}_{\text{height }0} \to \underbrace{K(\asymp)\{2\}}_{\text{height }1}\to 0\Big);$$

$$K(\times) = \text{Flatten}\Big(0\to \underbrace{K(\asymp)\{-2\}}_{\text{height }-1} \to \underbrace{K(\asymp)\{-1\}}_{\text{height }0}\to 0\Big);$$



$$\longrightarrow (V\otimes V\xrightarrow{m} V)\qquad m: \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$$

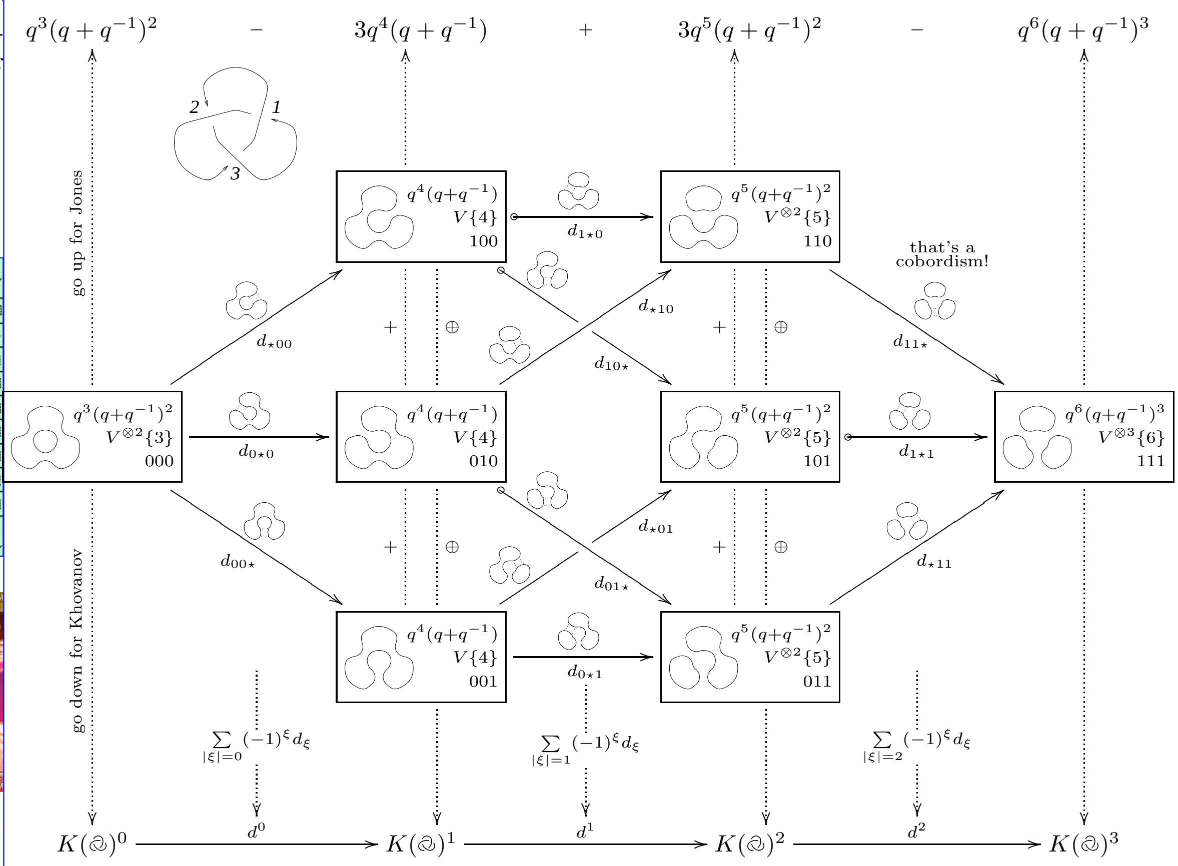


$$\longrightarrow (V\xrightarrow{\Delta} V\otimes V)\qquad \Delta: \begin{cases} v_+\mapsto v_+\otimes v_- + v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$$

Categorifying $\mathbb{Z}[q^{\pm 1}]$. $f=\sum a_j q^j$ becomes $V=\bigoplus V_j$ s.t. $q\dim V := \sum q^j\dim V_j = f$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$, $\deg d = 0$, and $\chi_q(V^\bullet) := \sum(-1)^r q\dim V^r = f = \sum(-1)^r q\dim H^r$.

Note. Setting $V\{l\}_j := V_{j-l}$, we get $q\dim V\{l\} = q^l\, q\dim V$.

Example: $= q+q^3+q^5-q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $= K(\cdot)$.

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\operatorname{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\operatorname{Kh}(L)$ is strictly stronger than $J(L)$: $J(\bar{5}_1) = J(10_{132})$ yet $\operatorname{Kh}(\bar{5}_1)\neq\operatorname{Kh}(10_{132})$.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my
http://www.math.toronto.edu/~drorbn/papers/Categorification/.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/