

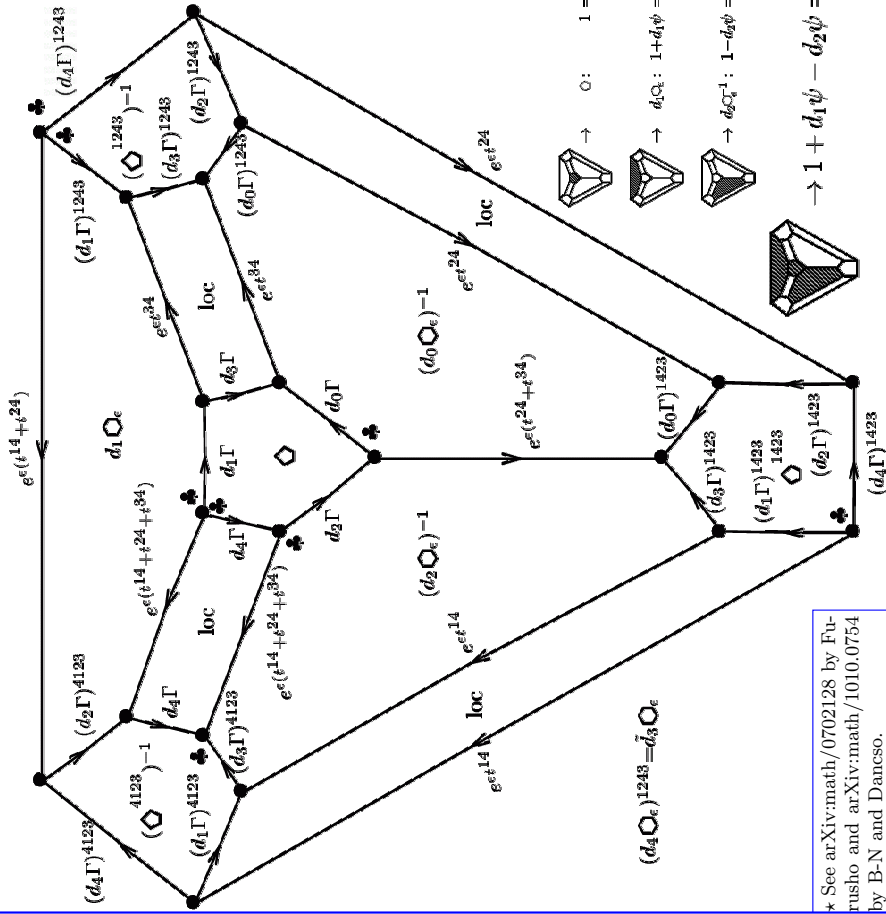
## Braids and the Grothendieck-Teichmuller Group

Dror Bar-Natan, the Newton Institute, January 2013. <http://www.math.toronto.edu/~drorbn/Talks/Newton-1301/>

**Abstract.** The “Grothendieck-Teichmuller Group” (GT) appears as a “depth certificate” in many recent works — we do  $A$  to  $B$ , apply the result to  $C$ , and get something related to GT, therefore it must be interesting”. Interesting or not, in my talk I will explain how GT arose first, in Drinfel’d’s work on associators, and how it can be used to show that “every bounded-degree associator extends”, that “rational associators exist”, and that “the pentagon implies the hexagon”\*.

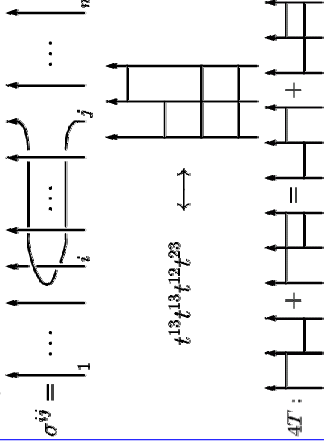
In a nutshell: the filtered tower of braid groups (with bells and whistles attached) is isomorphic to its associated graded, but the isomorphism is neither canonical nor unique — such an isomorphism is precisely the thing called “an associator”. But the set of isomorphisms between two isomorphic objects always has two groups acting simply transitively on it — the group of automorphisms of the first object acting on the right, and the group of automorphisms of the second object acting on the left. In the case of associators, that first group is what Drinfel’d calls the Grothendieck-Teichmuller group GT, and the second group, isomorphic but not canonically to the first and denoted GRT, is the one several recent works seem to refer to.

Almost everything I will talk about is in my old paper “On Associators and the Grothendieck-Teichmuller Group  $\Gamma$ ”, also at arXiv:q-alg/9606021.

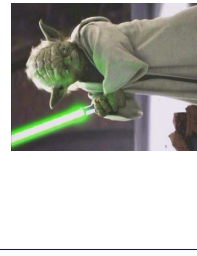


\* See arXiv:math/0702128 by Furusho and arXiv:math/1010.0754 by B-N and Danco.

**Baby(!) Example.**  $PB_n$ : pure braids;  $I \subset QPB_n$  the augmentation ideal;  $B^{(m)} = QPB_n/I^{m+1}$  (filtered!);  $B = \varinjlim B^{(m)}$  (filtered!). Then  $\text{gr } B^{(m)} = C^{(m)}$  and then  $\text{gr } \hat{B} = \hat{C}$  where  $C = [t^{ij}, t^{jk}, t^{ki}] = [t^{ij}, t^{jk} + t^{kj}] = 0$ , so  $B^{(m)}$  and  $B$  are isomorphic to  $C^{(m)}$  and  $\hat{C}$ , but not canonically. Me not know that the groups GT and GRT here have been analyzed.



**Main Theorem.** The projection  $\text{ASSO}^{(m)} \rightarrow \text{ASSO}^{(m-1)}$  is surjective. (yey!)  
**Sketch.** Given  $\text{ASSO}^{(m)} \neq \emptyset$  (hard, analytic), sufficient is surjectivity of  $\text{GRT}^{(m)} \rightarrow \text{GRT}^{(m-1)}$ , enough is surjectivity of  $\text{grt}^{(m)} \rightarrow \text{grt}^{(m-1)}$ , polygon on left use, little homological algebra too.



$$1 = d_1 \Gamma_{\text{loc}} \Gamma^{-1} (d_1 \Gamma)^{-1},$$

$$d_1 \text{Cl}: 1 + d_1 \psi = d_1 \Gamma e^{t^{14} + t^{24}} (d_1 \Gamma)^{-1} e^{t^{14} + t^{24}} (d_2 \Gamma)^{-1} e^{t^{14} + t^{24}} (d_2 \Gamma)^{-1} e^{t^{14} + t^{24}} (d_4 \Gamma)^{-1},$$

$$\rightarrow d_2 \text{Cl}^{-1}: 1 - d_2 \psi = (\text{product around shaded area}).$$

$$\rightarrow 1 + d_1 \psi - d_2 \psi = (\text{product around shaded area})$$

by successive approximations presents no problems. For this we introduce the following modification  $\text{GRT}_1(k)$  of the group  $\text{GT}(k)$ . We denote by  $\text{GRT}_1(k)$  the set of all  $g \in \text{Fr}_k(A, B)$  such that

$$g(B, A) = g(A, B)^{-1}, \quad (5.12)$$

$$g(C, A)g(B, C)g(A, B) = 1 \text{ for } A+B+C=0, \quad (5.13)$$

$$A+g(A, B)^{-1}g(A, C)+g(A, C)^{-1}g(A, C)=0 \text{ for } A+B+C=0, \quad (5.14)$$

$$g(X^{12}, X^{23} + X^{24})g(X^{13} + X^{23}, X^{34}) = g(X^{23}, X^{34})g(X^{12} + X^{13}, X^{24} + X^{34})g(X^{12}, X^{23}), \quad (5.15)$$

where the  $X^{ij}$  satisfy (5.1).  $\text{GRT}_1(k)$  is a group with the operation  $(g_1 \circ g_2)(A, B) = g_1(g_2(A, B)A)g_2(A, B)^{-1} \cdot g_2(A, B)$ . On  $\text{GRT}_1(k)$  there is an action of  $k^*$ , given by  $\tilde{g}(A, B) = g(c^{-1}A, c^{-1}B)$ ,  $c \in k^*$ . The semidirect product of  $k^*$  and  $\text{GRT}_1(k)$ , we denote by  $\text{GRT}(k)$ . The Lie algebra  $\text{grt}_1(k)$  of the group  $\text{GRT}_1(k)$  consists of the series  $\psi \in \text{Fr}_k(A, B)$  such that

$$\psi(B, A) = -\psi(A, B), \quad (5.17)$$

$$\psi(C, A) + \psi(B, C) + \psi(A, B) = 0 \text{ for } A+B+C=0, \quad (5.18)$$

$$[B, \psi(A, B)] + [C, \psi(A, C)] = 0 \text{ for } A+B+C=0, \quad (5.19)$$

$$\psi(X^{12}, X^{23} + X^{24}) + \psi(X^{13} + X^{23}, X^{34}) = \psi(X^{23}, X^{34}) + \psi(X^{12} + X^{13}, X^{24} + X^{34}) + \psi(X^{12}, X^{23}), \quad (5.20)$$

where the  $X^{ij}$  satisfy (5.1). A commutator  $(\cdot, \cdot)$  in  $\text{grt}_1(k)$  is of the form  $(\psi_1, \psi_2) = [\psi_1, \psi_2] + D_{\psi_1}(\psi_2) - D_{\psi_2}(\psi_1)$ .

where  $[\psi_1, \psi_2]$  is the commutator in  $\text{Fr}_k(A, B)$  and  $D_{\psi}$  is the derivation of  $\text{Fr}_k(A, B)$  given by  $D_{\psi}(A) = [\psi, A]$ ,  $D_{\psi}(B) = 0$ . The algebra  $\text{grt}_1(k)$  is

PROPOSITION 5.1. The action of  $\text{GT}(k)$  on  $M(k)$  is free and transitive.

PROOF. If  $(\mu, \varphi) \in M(k)$  and  $(\bar{\mu}, \bar{\varphi}) \in M(k)$ , then there is exactly one  $f$  such that  $\bar{\varphi}(A, B) = f(\varphi(A, B)e^{\lambda(A, B)^{-1}}, e^{\lambda(A, B)})$ . We need to show that  $(\lambda, f) \in \text{GT}(k)$ , where  $\lambda = \bar{\mu}/\mu$ . We prove (4.10). Let  $G_n$  be the semidirect product of  $S_n$  and  $\text{exp } \mathfrak{a}_n^k$ . Consider the homomorphism  $B_n \rightarrow G_n$  that takes  $\sigma_i$  into

$$\varphi(X^{ij} + \dots + X^{i-1, j}, X^{i, i+1})^{-1} \sigma_i^{-1} e^{\mu X^{i, i+1}/2} \varphi(X^{i, i+1} + \dots + X^{i-1, j}, X^{i, i+1}),$$

where  $\sigma^j \in S_n$  transposes  $i$  and  $j$ . It induces a homomorphism  $K_n \rightarrow \text{exp } \mathfrak{a}_n^k$  and therefore a homomorphism  $\alpha_n: K_n(k) \rightarrow \text{exp } \mathfrak{a}_n^k$ , where  $K_n(k)$  is the  $k$ -pro-unipotent completion of  $K_n$ . It is easily shown that the left- and right-hand sides of (4.10) have the same images in  $\text{exp } \mathfrak{a}_n^k$ . It remains to prove that  $\alpha_n$  is an isomorphism. The algebra  $\text{Lie } K_n(k)$  is topologically generated by the elements  $\xi_{ij}$ ,  $1 \leq i < j \leq n$ , with defining relations obtained from (4.7)–(4.9) by substituting  $x_{ij} = \text{exp } \xi_{ij}$ . The principal parts of these relations are the same as in (5.1), while  $(\alpha_n)_*(\xi_{ij}) = \mu X^{ij} + \{\text{lower terms}\}$ , where  $(\alpha_n)_*$ :  $\text{Lie } K_n(k) \rightarrow \mathfrak{a}_n^k$  is induced by the homomorphism  $\alpha_n$ . Therefore  $\alpha_n$  is an isomorphism, i.e., (4.10) is proved. (4.3) is obvious. To prove (4.4), we can interpret it in terms of  $K_3$  and argue as in the proof of (4.10), or, what is equivalent, make the substitution

$$X_1 = e^A, \quad X_2 = e^{-A/2} \varphi(B, A) e^B \varphi(B, A)^{-1} e^{A/2},$$

$$X_3 = \varphi(C, A) e^C \varphi(C, A)^{-1},$$

where  $A+B+C=0$ .



From Drinfel'd's "On quasitriangular Quasi-Hopf algebras and a group closely connected with  $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ ", Leningrad Math. J. 2 (1991) 829–860.