

Balloons and Hoops and their Universal Finite-Type Invariant, BF Theory, and an Ultimate Alexander Invariant

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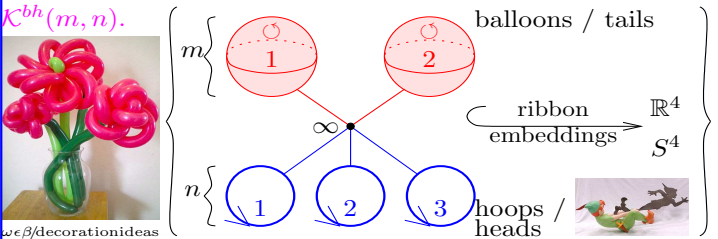
$\omega \in \beta := \text{http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121}$



Scheme. • Balloons and hoops in \mathbb{R}^4 , algebraic structure and relations with 3D.

• An ansatz for a “homomorphic” invariant: computable, related to finite-type and to BF.

• Reduction to an “ultimate Alexander invariant”.



Examples.

ϵ_x :

ϵ_u :

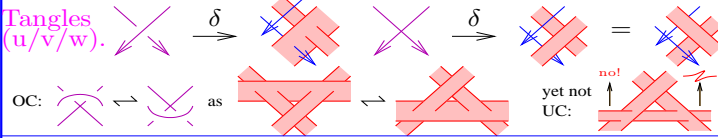
ρ_{ux}^+ :

ρ_{ux}^- :

I mean business!

$T_0 = \text{Rm}[3, a] \text{Rp}[2, 2] \text{Rp}[1, 4]$
 $S = T_0 // \text{dm}[2, 1, 1] // \text{dm}[4, b, b] //$
 $\text{dm}[1, a, a] // \text{dm}[3, a, a]$
 $S[[\{S}]] / \cdot (w_{CW} \mapsto (\text{Deg}[w] + 1) \cdot w, w_{CW} \mapsto \text{Deg}[w] \cdot w)$

$\mu[\text{CW}[1, a], -2 \text{[ab]}, -2 \text{[ab]}, -3 \text{[aab]} - 3 \text{[abb]}, -4 \text{[aaab]} - 42 \text{[aabb]} - 60 \text{[abab]} - 4 \text{[abbb]}, -5 \text{[aaaab]} - 110 \text{[aaabb]} - 180 \text{[aabbab]} - 110 \text{[aabbba]} - 180 \text{[ababbb]} - 5 \text{[abbbb]}, h[b] \text{LS}[2(a), 0, -24 \text{[aab]}, -60 \text{[aaab]} + 60 \text{[aabb]}, -120 \text{[aaaab]} + 900 \text{[aaabb]} + 360 \text{[aabbab]} - 120 \text{[aabbba]} + h[a] \text{LS}[-2(a) + 2(b), 9 \text{[ab]}, 26 \text{[aabb]} - 26 \text{[abb]}, 60 \text{[aaab]} - 255 \text{[aabb]} - 60 \text{[abbb]}, 119 \text{[aaaab]} - 1504 \text{[aaabb]} - 118 \text{[aabbab]} + 1504 \text{[aabbba]} - 1386 \text{[ababbb]} - 119 \text{[abbbb]]]$



• δ injects u-Knots into \mathcal{K}^{bh} (likely u-tangles too).

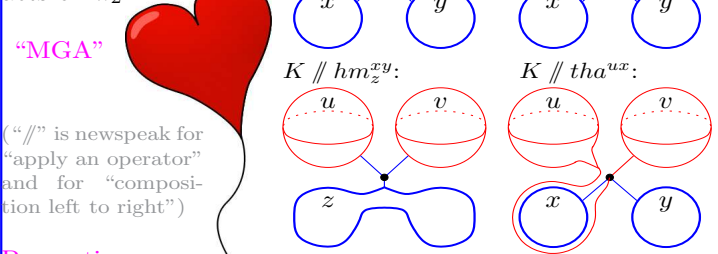
• δ maps v/w-tangles map to \mathcal{K}^{bh} ; the kernel contains Reidemeister moves and the “overcrossings commute” relation, and **conjecturally**, that’s all. Allowing punctures and cuts, δ is onto.

Operations
Punctures & Cuts

Connected Sums:

Meta-Group-Action. K :

If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

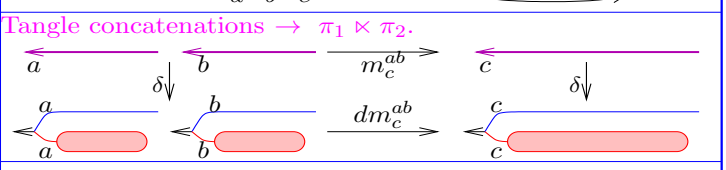


Properties.

- Associativities: $m_a^{ab} // m_a^{ac} = m_b^{bc} // m_a^{ab}$, for $m = tm, hm$.
- Action axiom t : $tm_w^{uv} // tha^{wx} = tha^{ux} // tha^{vx} // tm_w^{uv}$,
- Action axiom h : $hm_z^{xy} // tha^{uz} = tha^{ux} // tha^{uy} // hm_z^{xy}$.
- SD Product: $dm_c^{ab} := tha^{ab} // tm_c^{ab} // hm_c^{ab}$ is associative.

Meta-associativity.

$m_a^{ab} // m_a^{ac} = m_b^{bc} // m_a^{ab}$



Thus we seek homomorphic invariants of \mathcal{K}^{bh} !

Invariant #0. With Π_1 denoting “honest π_1 ”, map $\gamma \in \mathcal{K}^{bh}(m, n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1 , and the “longitudes” x_j are some elements of Π_1 . $*$ acts like $*$, tm acts by “merging” two meridians/generators, hm acts by multiplying two longitudes, and tha^{ux} acts by “conjugating a meridian by a longitude”:

$(\Pi, (u, \dots), (x, \dots)) \mapsto (\Pi * \langle \bar{u} \rangle / (u = x\bar{u}x^{-1}), (\bar{u}, \dots), (x, \dots))$

Failure #0. Can we write the x ’s as free words in the u ’s?

If $x = uv$, compute $x // tha^{ux}$:

$$x = uv \rightarrow \bar{u}v = u^xv = u^{\bar{u}v}v = u^{u^xv}v = u^{u^{u^xv}v}v = \dots$$

The Meta-Group-Action M . Let T be a set of “tail labels” (“balloon colours”), and H a set of “head labels” (“hoop colours”). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T , so there’s $\text{tr} : FA \rightarrow CW$. Let $M(T, H) := \{(\bar{\lambda} = (x : \lambda_x)_{x \in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$

$$= \left\{ \left(x : \begin{array}{c} u \\ \diagdown \\ v \end{array}, y : \begin{array}{c} v \\ \diagup \\ u \end{array} \right) ; \begin{array}{c} u \\ \diagdown \\ v \end{array} \begin{array}{c} u \\ \diagup \\ v \end{array} \right\} \dots$$

Operations. Set $(\bar{\lambda}_1; \omega_1) * (\bar{\lambda}_2; \omega_2) := (\bar{\lambda}_1 \cup \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}; \omega)$ define

$$tm_w^{uv} : \mu \mapsto \mu // (u, v \mapsto w),$$

$$hm_z^{xy} : \mu \mapsto \left((\dots, \widehat{x : \lambda_x}, \widehat{y : \lambda_y}, \dots, z : \text{bch}(\lambda_x, \lambda_y)) ; \omega \right)$$

“stable apply”

$$tha^{ux} : \mu \mapsto \underbrace{\mu // (u \mapsto e^{\text{ad } \lambda_x}(\bar{u}))}_{\mu // CC_u^\lambda} // (\bar{u} \mapsto u) + (0; J_u(\lambda_x))$$

the “ J -spice”

A CC_u^λ example.

