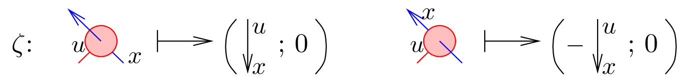
Trees and Wheels and Balloons and Hoops: Why I Care

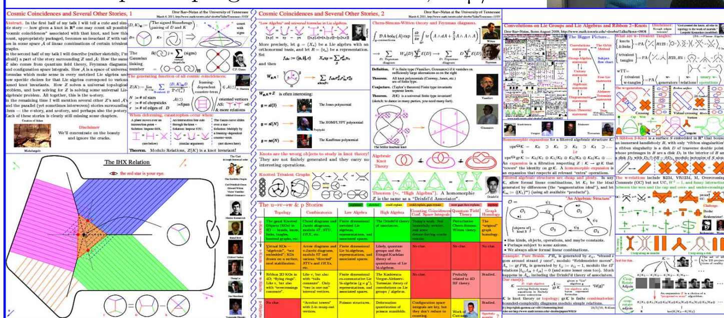
Moral. To construct an M-valued invariant ζ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare ζ on the generators, and verify the relations that δ satisfies.

The Invariant ζ. Set $\zeta(\epsilon_x) = (x \to 0; 0)$, $\zeta(\epsilon_u) = ((); 0)$, and



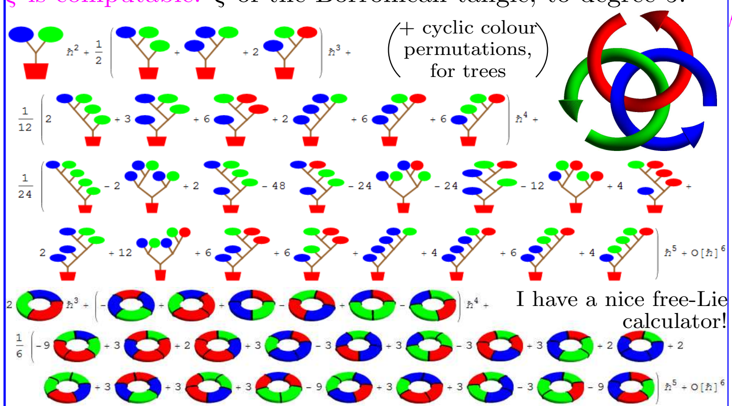
Theorem. ζ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$. (... and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko



See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

ζ is computable! ζ of the Borromean tangle, to degree 5:



Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \mathrm{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T;H) \to \mathrm{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

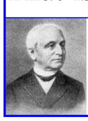
ζ and BF Theory. (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\mathrm{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be (roughly) the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B \wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \mathrm{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

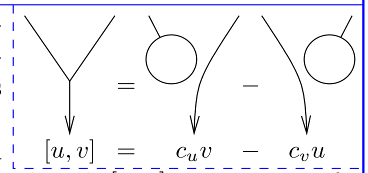
That is, ζ is a complete evaluation of the BF TQFT.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified) www.katlas.org

The β quotient is M divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,



$$\mu \to ((\lambda_x); \omega) \quad \text{with } \lambda_x = \sum_{u\in T} \lambda_{ux} u x, \quad \lambda_{ux}, \omega \in R,$$

$$\mathrm{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u}-1}{c_u} u + e^{c_u}\frac{e^{c_v}-1}{c_v} v\right),$$

if $\gamma = \sum \gamma_v v$ then with $c_\gamma := \sum \gamma_v c_v$,

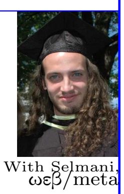
$$u /\!\!/ RC_u^\gamma = \left(1 + c_u\gamma_u \frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma} u - c_u \frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_v v\right),$$

$\mathrm{div}_u \gamma = c_u \gamma_u$, and $J_u(\gamma) = \log\left(1 + \frac{e^{c_\gamma}-1}{c_\gamma} c_u \gamma_u\right)$, so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x \to \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x}-1}{c_x}$ and $\omega \to e^\omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(T;H)$ be

$$\left\{\begin{array}{c|ccc|l} \omega & x & y & \cdots & \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot & \text{rational functions in} \\ v & \alpha_{vx} & \alpha_{vy} & \cdot & \text{variables } t_u, \text{ one for} \\ \vdots & \cdot & \cdot & \cdot & \text{each } u \in T. \end{array}\right\},$$



With Selmani, ωεβ/meta

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} * \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|cccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

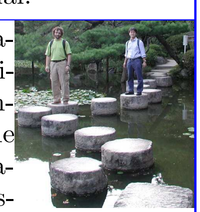
$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why happy? An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*



See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

May class: ωεβ/aarhus Class next year: ωεβ/1350

Paper: ωεβ/kbh

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Geneva-130917/, .../Toronto-1303/ and at .../Chicago-1303/