

Trees and Wheels and Balloons and Hoops

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$\omega\epsilon\beta := \text{http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919}$



15 Minutes on Algebra

Let T be a finite set of “tail labels” and H a finite set of “head labels”. Set

$$M_{1/2}(T; H) := FL(T)^H,$$

“ H -labeled lists of elements of the degree-completed free Lie algebra generated by T ”.

$$FL(T) = \left\{ 2t_2 - \frac{1}{2}[t_1, [t_1, t_2]] + \dots \right\} / \left(\begin{array}{c} \text{anti-symmetry} \\ \text{Jacobi} \end{array} \right) \dots \text{with the obvious bracket.}$$

$$M_{1/2}(u, v; x, y) = \left\{ \lambda = \left(x \rightarrow \begin{array}{c} u \\ \swarrow \searrow \\ v \end{array}, y \rightarrow \begin{array}{c} v \\ \swarrow \searrow \\ u \end{array} - \frac{22}{7} \begin{array}{c} u \\ \swarrow \searrow \\ v \end{array} \right) \dots \right\}$$

Operations $M_{1/2} \rightarrow M_{1/2}$.

Tail Multiply tm_w^{uv} is $\lambda \mapsto \lambda \parallel (u, v \rightarrow w)$, satisfies “meta-associativity”, $tm_u^{uv} \parallel tm_u^{uv} = tm_v^{uv} \parallel tm_u^{uv}$.

Head Multiply hm_z^{xy} is $\lambda \mapsto (\lambda \setminus \{x, y\}) \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y))$, where

$$\text{bch}(\alpha, \beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha, \beta]}{2} + \frac{[\alpha, [\alpha, \beta]] + [[\alpha, \beta], \beta]}{12} + \dots$$

satisfies $\text{bch}(\text{bch}(\alpha, \beta), \gamma) = \log(e^{\alpha} e^{\beta} e^{\gamma}) = \text{bch}(\alpha, \text{bch}(\beta, \gamma))$ and hence meta-associativity, $hm_x^{xy} \parallel hm_x^{xz} = hm_y^{yz} \parallel hm_x^{xy}$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda \parallel RC_u^{\lambda_x}$, where $C_u^{-\gamma}: FL \rightarrow FL$ is the substitution $u \rightarrow e^{-\gamma} u e^{\gamma}$, or more precisely,

$$C_u^{-\gamma}: u \rightarrow e^{-\text{ad} \gamma}(u) = u - [\gamma, u] + \frac{1}{2}[\gamma, [\gamma, u]] - \dots,$$

and $RC_u^\gamma = (C_u^{-\gamma})^{-1}$. Then $C_u^{\text{bch}(\alpha, \beta)} = C_u^\alpha \parallel RC_u^{-\beta} \parallel C_u^\beta$ hence $RC_u^{\text{bch}(\alpha, \beta)} = RC_u^\alpha \parallel RC_u^\beta \parallel RC_u^\alpha$ hence “meta $u^{xy} = (u^x)^y$ ”,

$$hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy},$$

and $tm_w^{uv} \parallel C_w^\gamma \parallel tm_w^{uv} = C_w^\gamma \parallel RC_w^{-\gamma} \parallel C_w^\gamma \parallel tm_w^{uv}$ and hence “meta $(uv)^x = u^x v^x$ ”, $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$.

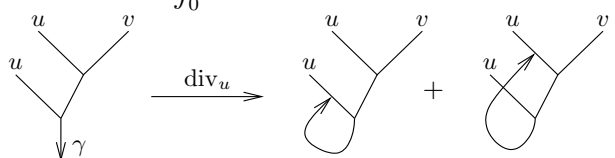
Wheels. Let $M(T; H) := M_{1/2}(T; H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T , or equally well, on $FL(T)$:



Operations. On $M(T; H)$, define tm_w^{uv} and hm_z^{xy} as before, and tha^{ux} by adding some J -spice:

$$(\lambda; \omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x},$$

where $J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}$, and



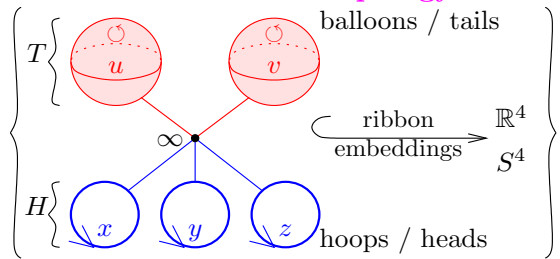
Theorem Blue. All blue identities still hold.

Merge Operation. $(\lambda_1; \omega_1) * (\lambda_2; \omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

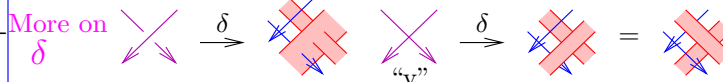
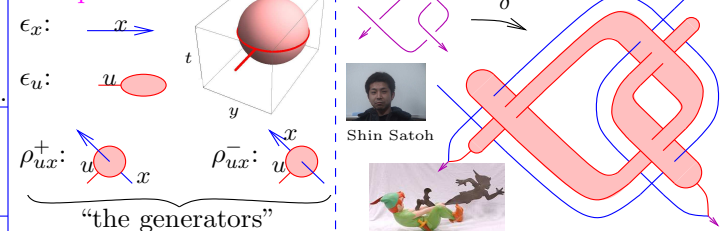
$\mathcal{K}^{bh}(T; H)$.

“Ribbon-knotted balloons and hoops”

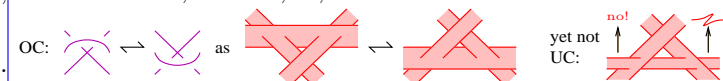
15 Minutes on Topology



Examples.



satisfies R123, VR123, D, and



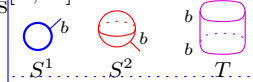
- δ injects u-knots into \mathcal{K}^{bh} (likely u-tangles too).
- δ maps v-tangles to \mathcal{K}^{bh} ; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

Operations

Punctures & Cuts **Connected Sums.**

If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

Riddle. People often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$. Why not $\pi_T(X) := [T, X]$?

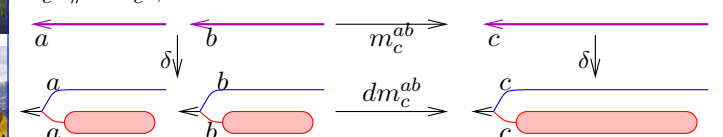


“Meta-Group-Action”

Properties.

- Associativities: $m_a^{ab} \parallel m_a^{ac} = m_b^{bc} \parallel m_a^{ab}$, for $m = tm, hm$.
- “(uv) $^x = u^x v^x$ ”: $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$.
- “($u(xy) = (u^x)^y$ ”): $hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy}$.

Tangle concatenations $\rightarrow \pi_1 \times \pi_2$. With $dm_c^{ab} := tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}$,



Finite type invariants make sense in the usual way, and “algebra” is (the primitive part of) “gr” of “topology”.