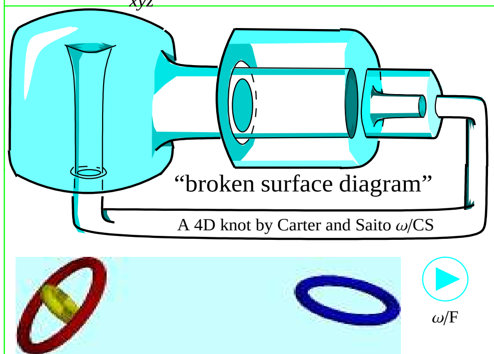
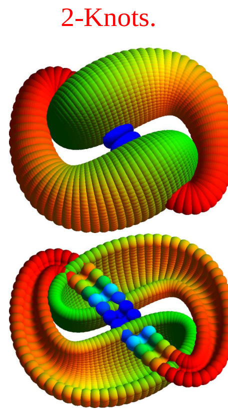
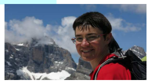
Knots in Four Dimensions and the Simplest Open Problem About Them

Abstract. I will describe a few 2-dimensional knots in 4 dimensional space in detail, then tell you how to make many more, then tell you that I don't really understand my way of making them, yet I can tell at least some of them apart in a colourful way.

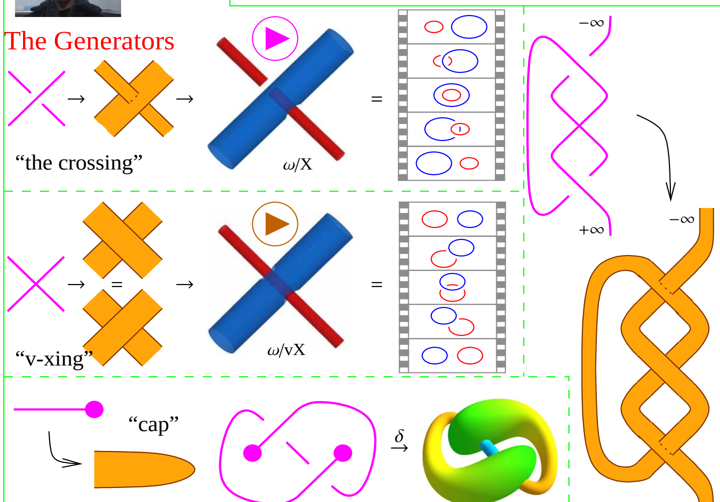
u-Knots.





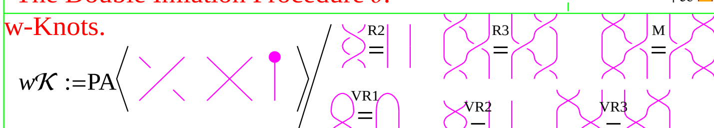


The Generators

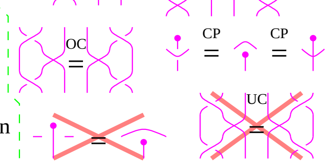


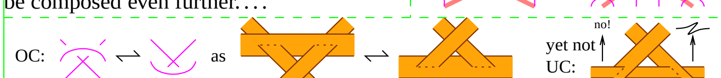
The Double Inflation Procedure δ.

w-Knots.



"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles, which can then be composed even further. . . .





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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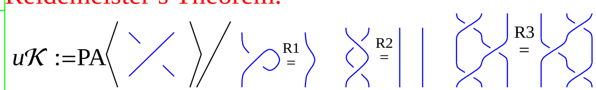
Satoh's Conjecture. (ω/Sat) The "kernel" of the double inflation map δ, mapping w-knot diagrams in the plane to knotted 2D tubes and spheres in 4D, is precisely the moves R2-3, VR1-3, M, CP and OC listed above. In other words, two w-knot diagrams represent via δ the same 2D knot in 4D iff they differ by a sequence of the said moves.

First Isomorphism Thm: $\delta\colon G \to H \;\Rightarrow\; \operatorname{im}\delta \cong G/\ker(\delta)$

δ is a map from algebra to topology. So a thing in "hard" topology (im δ) is the same as a thing in "easy" algebra ($w\mathcal{K}$).

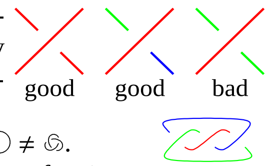
Reidemeister's Theorem.



Proof by a genericity / "shaking" argument

Kurt Reidemeister

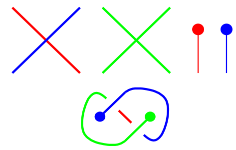
3-Colourings. Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic; $\lambda(K) := |\{\text{3-colourings}\}|$.

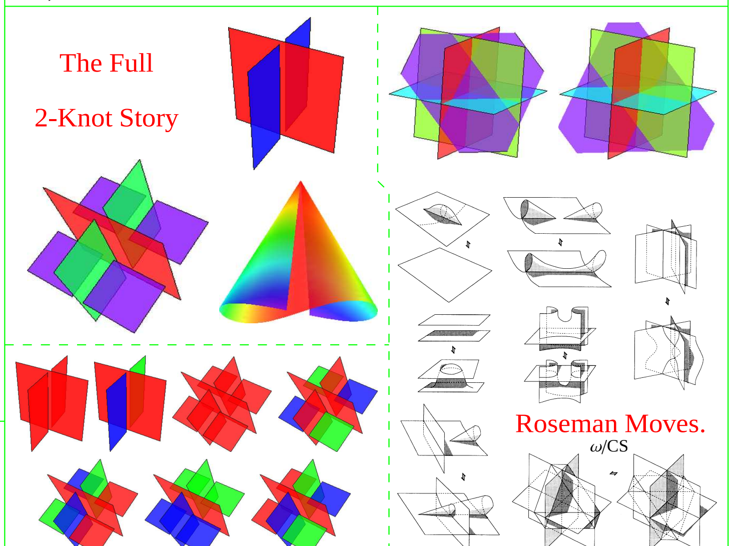


Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{trefoil}) = 9$; so $\bigcirc \neq$ trefoil.

Exercise. Show that the set of colourings of K is a vector space over $\mathbb{F}_3$ hence $\lambda(K)$ is always a power of 3.

Extend λ to $w\mathcal{K}$ by declaring that arcs "don't see" v-xings, and that caps are always "kosher". Then $\lambda(\bullet\!\!-\!\!\bullet) = 3 \neq 9 = \lambda(\text{CS 2-knot})$, so assuming Conjecture, the CS 2-knot is indeed knotted.





Expansions. Given a "ring" K and an ideal $I \subset K$, set

$$A := I^0/I^1 \oplus I^1/I^2 \oplus I^2/I^3 \oplus \cdots.$$

A homomorphic expansion is a multiplicative $Z\colon K \to A$ such that if $\gamma \in I^m$, then $Z(\gamma) = (0, 0, \ldots, 0, \gamma/I^{m+1}, *, *, \ldots)$.

Example. Let $K = C^\infty(\mathbb{R}^n)$ be smooth functions on $\mathbb{R}^n$, and $I := \{f \in K\colon f(0) = 0\}$. Then $I^m = \{f\colon f \text{ vanishes as } |x|^m\}$ and I^m/I^{m+1} is {homogeneous polynomials of degree m} and A is the set of power series. So Z is "a Taylor expansion".

Hence Taylor expansions are vastly general; even knots can be Taylor expanded!