Let x_c denote the path on which $\mathcal{L}(x)$ attains its minimum value, write $x = x_c + x_q$ with $x_q \in W_{00}$, and get

$$\psi(T, x) = c \int dx_0 \psi_0(x_0) \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c + x_q)}.$$

In our particular case $\mathcal{L}$ is quadratic in x, and therefore $\mathcal{L}(x_c + x_q) = \mathcal{L}(x_c) + \mathcal{L}(x_q)$ (this uses the fact that x_c is an extremal of $\mathcal{L}$, of course). Plugging this into what we already have, we get

$$\begin{aligned}\psi(T, x) &= c \int dx_0 \psi_0(x_0) \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_c) + i\mathcal{L}(x_q)} \\ &= c \int dx_0 \psi_0(x_0) e^{i\mathcal{L}(x_c)} \int_{W_{00}} \mathcal{D}x_q e^{i\mathcal{L}(x_q)}.\end{aligned}$$

Now this is excellent news, because the remaining path integral over W_{00} does not depend on x_0 or x_n, and hence it is a constant! Allowing c to change its value from line to line, we get

$$\psi(T, x) = c \int dx_0 \psi_0(x_0) e^{i\mathcal{L}(x_c)}.$$

Lemma 3.4 now shows us that $x_c(t) = x_0 \cos t + x_n \sin t$. An easy explicit computation gives $\mathcal{L}(x_c) = -x_0 x_n$, and we arrive at our final result,

$$\psi(\frac{\pi}{2}, x) = c \int dx_0 \psi_0(x_0) e^{-ix_0 x_n}.$$

Notice that this is precisely the formula for the Fourier transform of ψ_0! That is, the answer to the question in the title of this document is "the particle gets Fourier transformed", whatever that may mean.

3. The Lemmas

Lemma 3.1. *For any two matrices A and B,*

$$e^{A+B} = \lim_{n\to\infty} \left(e^{A/n} e^{B/n}\right)^n.$$

Proof. (sketch) Using Taylor expansions, we see that $e^{\frac{A+B}{n}}$ and $e^{A/n}e^{B/n}$ differ by terms at most proportional to c/n^2. Raising to the nth power, the two sides differ by at most $O(1/n)$, and thus

$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A+B}{n}}\right)^n = \lim_{n\to\infty} \left(e^{A/n} e^{B/n}\right)^n,$$

as required. □

Lemma 3.2.

$$\left(e^{itV}\psi_0\right)(x) = e^{itV(x)}\psi_0(x).$$

□

Lemma 3.3.

$$\left(e^{i\frac{t}{2}\Delta}\psi_0\right)(x) = c \int dx' e^{i\frac{(x-x')^2}{2t}} \psi_0(x').$$

Proof. In fact, the left hand side of this equality is just a solution $\psi(t, x)$ of Schrödinger's equation with $V = 0$:

$$\frac{\partial \psi}{\partial t} = \frac{i}{2}\Delta_x \psi, \qquad \psi|_{t=0} = \psi_0.$$

Taking the Fourier transform $\tilde{\psi}(t, p) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx}\psi(t, x)dx$, we get the equation

$$\frac{\partial \tilde{\psi}}{\partial t} = -i\frac{p^2}{2}\tilde{\psi}, \qquad \tilde{\psi}|_{t=0} = \tilde{\psi}_0.$$

For a fixed p, this is a simple first order linear differential equation with respect to t, and thus,

$$\tilde{\psi}(t, p) = e^{-i\frac{tp^2}{2}}\tilde{\psi}_0(p).$$

Taking the inverse Fourier transform, which takes products to convolutions and Gaussians to other Gaussians, we get what we wanted to prove. □

Lemma 3.4. *With the notation of Section 2 and at the specific case of $V(x) = \frac{1}{2}x^2$ and $T = \frac{\pi}{2}$, we have*

$$x_c(t) = x_0 \cos t + x_n \sin t.$$

Proof. If x_c is a critical point of $\mathcal{L}$ on $W_{x_0 x_n}$, then for any $x_q \in W_{00}$ there should be no term in $\mathcal{L}(x_c + \epsilon x_q)$ which is linear in ϵ. Now recall that

$$\mathcal{L}(x) = \int_0^T dt \left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right),$$

so using $V(x_c + \epsilon x_q) \sim V(x_c) + \epsilon x_q V'(x_c)$ we find that the linear term in ϵ in $\mathcal{L}(x_c + \epsilon x_q)$ is

$$\int_0^T dt \left(\dot{x}_c \dot{x}_q - V'(x_c)x_q\right).$$

Integrating by parts and using $x_q(0) = x_q(T) = 0$, this becomes

$$\int_0^T dt \left(-\ddot{x}_c - V'(x_c)\right) x_q.$$

For this integral to vanish independently of x_q, we must have $-\ddot{x}_c - V'(x_c) \equiv 0$, or

$$\ddot{x}_c = -V'(x_c).$$

(This is the famous $F = ma$ of Newton's, and we have just rediscovered the principle of least action!)

In our particular case this boils down to the equation

$$\ddot{x}_c = -x_c, \qquad x_c(0) = x_0, \qquad x_c(\pi/2) = x_n,$$

whose unique solution is displayed in the statement of this lemma. □