What happens to a quantum particle on a pendulum at $T = \frac{\pi}{2}$?

Abstract. This subject is the best one-hour introduction I know for the mathematical techniques that appear in quantum mechanics — in one short lecture we start with a meaningful question, visit Schrödinger's equation, operators and exponentiation of operators, Fourier analysis, path integrals, the least action principle, and Gaussian integration, and at the end we land with a meaningful and interesting answer.

Based a lecture given by the author in the "trivial notions" seminar in Harvard on April 29, 1989. This edition, January 10, 2014.

1. The Question

Let the complex valued function $\psi = \psi(t, x)$ be a solution of the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -i\left(-\frac{1}{2}\Delta_x + \frac{1}{2}x^2\right)\psi \qquad \text{with} \qquad \psi|_{t=0} = \psi_0.$$

What is $\psi|_{t=T=\frac{\pi}{2}}$?

In fact, the major part of our discussion will work just as well for the general Schrödinger equation,

$$\frac{\partial\psi}{\partial t} = -iH\psi, \qquad H = -\frac{1}{2}\Delta_x + V(x),$$

$$\psi|_{t=0} = \psi_0, \qquad \text{arbitrary } T,$$

where,

- ψ is the "wave function", with $|\psi(t, x)|^2$ representing the probability of finding our particle at time t in position x.
- H is the "energy", or the "Hamiltonian".
- $-\frac{1}{2}\Delta_x$ is the "kinetic energy".
- $V(x)$ is the "potential energy at x".

2. The Solution

The equation $\frac{\partial\psi}{\partial t} = -iH\psi$ with $\psi|_{t=0} = \psi_0$ formally implies

$$\psi(T, x) = \left(e^{-iTH}\psi_0\right)(x) = \left(e^{i\frac{T}{2}\Delta - iTV}\psi_0\right)(x).$$

By Lemma 3.1 with $n = 10^{58}+17$ and setting $x_n = x$ we find that $\psi(T, x)$ is

$$\left(e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\dots e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\psi_0\right)(x_n).$$

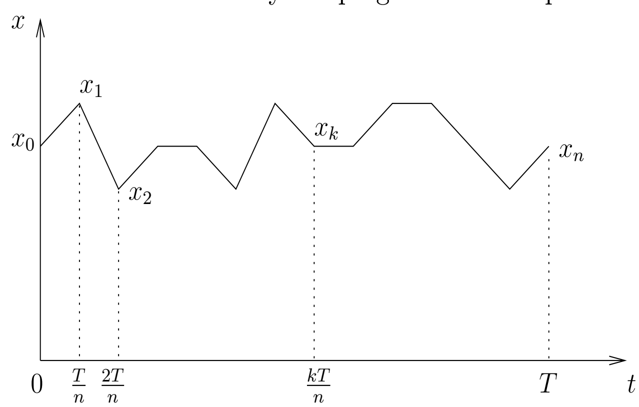
Now using Lemmas 3.2 and 3.3 we find that this is: (c denotes the ever-changing universal fixed numerical constant)

$$c\int dx_{n-1}e^{i\frac{(x_n-x_{n-1})^2}{2T/n}}e^{-i\frac{T}{N}V(x_{n-1})}\dots$$
$$\int dx_1 e^{i\frac{(x_2-x_1)^2}{2T/n}}e^{-i\frac{T}{N}V(x_1)}$$
$$\int dx_0 e^{i\frac{(x_1-x_0)^2}{2T/n}}e^{-i\frac{T}{N}V(x_0)}\psi_0(x_0).$$

Repackaging, we get

$$c\int dx_0\dots dx_{n-1}$$
$$\exp\left(i\frac{T}{2n}\sum_{k=1}^{n}\left(\frac{x_k-x_{k-1}}{T/n}\right)^2 - i\frac{T}{n}\sum_{k=0}^{n-1}V(x_k)\right)$$
$$\psi_0(x_0).$$

Now comes the novelty. keeping in mind the picture



and replacing Riemann sums by integrals, we can write

$$\psi(T, x) = c\int dx_0 \int_{W_{x_0x_n}}\mathcal{D}x$$
$$\exp\left(i\int_0^T dt\left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right)\right)\psi_0(x_0),$$

where $W_{x_0x_n}$ denotes the space of paths that begin at x_0 and end at x_n,

$$W_{x_0x_n} = \{x : [0, T] \to \mathbb{R} :\ x(0) = x_0,\ x(T) = x_n\},$$

and $\mathcal{D}x$ is the formal "path integral measure".

This is a good time to introduce the "action" $\mathcal{L}$:

$$\mathcal{L}(x) := \int_0^T dt\left(\frac{1}{2}\dot{x}^2(t) - V(x(t))\right).$$

With this notation,

$$\psi(T, x) = c\int dx_0\psi_0(x_0)\int_{W_{x_0x_n}}\mathcal{D}xe^{i\mathcal{L}(x)}.$$