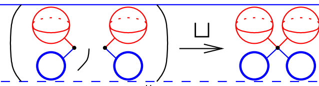
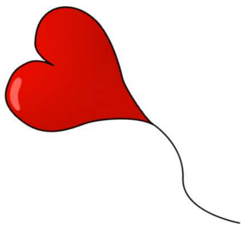
Some very good formulas for the Alexander polynomial, 2

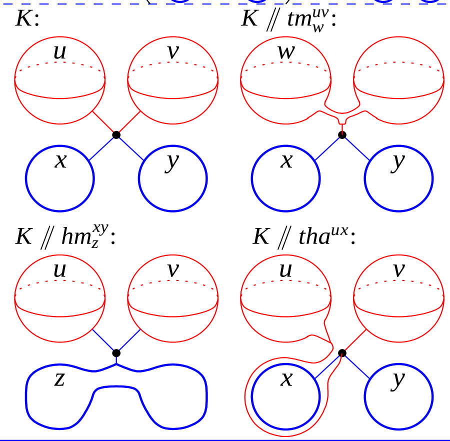
Operations
Punctures & Cuts | Connected Sums.



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Proposition. The generators generate.





Definition. l_{xu} is the linking number of hoop x with balloon u. For $x \in H$, $\sigma_x := \prod_{u\in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a\in T})$, the ring of rational functions in T variables.

Theorem 2 [BNS]. $\exists!$ an invariant $\beta\colon w\mathcal{K}^{bh}(H;T) \to R \times M_{T\times H}(R)$, intertwining

1. $$\left(\begin{array}{c|c}\omega_1 & H_1\\ \hline T_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & H_2\\ \hline T_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & H_1 & H_2\\ \hline T_1 & A_1 & 0\\ T_2 & 0 & A_2\end{array},$$

2. $$\begin{array}{c|c}\omega & H\\ \hline u & \alpha\\ v & \beta\\ T & \Xi\end{array} \xrightarrow{tm^{uv}_w} \left(\begin{array}{c|c}\omega & H\\ \hline w & \alpha+\beta\\ T & \Xi\end{array}\right)_{T_u,T_v\to T_w},$$

3. $$\begin{array}{c|ccc}\omega & x & y & H\\ \hline T & \alpha & \beta & \Xi\end{array} \xrightarrow{hm^{xy}_z} \begin{array}{c|cc}\omega & z & H\\ \hline T & \alpha+\sigma_x\beta & \Xi\end{array},$$

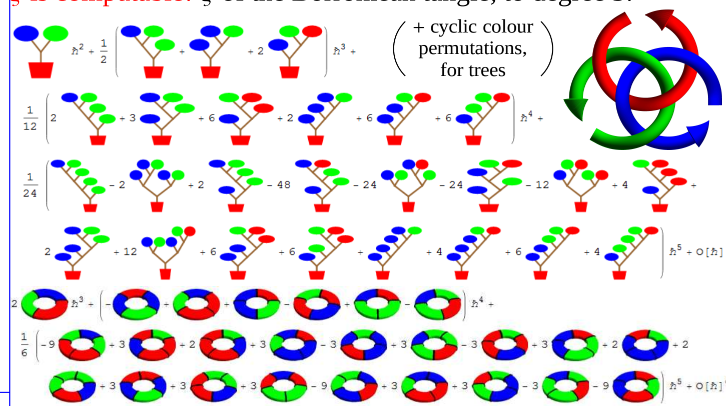
4. $$\begin{array}{c|cc}\omega & x & H\\ \hline u & \alpha & \theta\\ T & \phi & \Xi\end{array} \xrightarrow[\nu:=1+\alpha]{tha^{ux}} \begin{array}{c|cc}\nu\omega & x & H\\ \hline u & \sigma_x\alpha/\nu & \sigma_x\theta/\nu\\ T & \phi/\nu & \Xi-\phi\theta/\nu\end{array},$$

and satisfying $(\epsilon_x;\epsilon_u;\rho^{\pm}_{ux}) \xrightarrow{\beta} \left(\begin{array}{c|c}1 & x\\ \hline & \end{array}; \begin{array}{c|c}1 & \\ \hline u & \end{array}; \begin{array}{c|c}1 & x\\ \hline u & T_u^{\pm1}-1\end{array}\right)$.

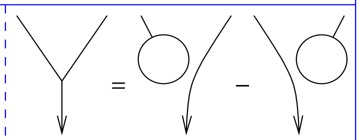
Proposition. If T is a u-tangle and $\beta(\delta T) = (\omega, A)$, then $\gamma(T) = (\omega, \sigma - A)$, where $\sigma = \operatorname{diag}(\sigma_a)_{a\in S}$. Under this, $m^{ab}_c \leftrightarrow tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$.

Theorem 3 [BND, BN]. $\exists!$ a homomorphic expansion, aka a homomorphic universal finite type invariant Z of w-knotted balloons and hoops. $\zeta := \log Z$ takes values in $FL(T)^H \times CW(T)$.

ζ is computable! ζ of the Borromean tangle, to degree 5:



Proposition [BN]. Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable, ζ reduces to β and the KBH operations on ζ reduce to the formulas in Theorem 2.

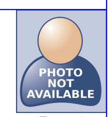


$[u, v] = c_u v - c_v u$

A Big Question. Does it all extend to arbitrary 2-knots (not necessarily "simple")? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(M, \mathfrak{g}^*)$,

$$S(A, B) := \int_M \langle B, F_A\rangle.$$



Rossi

With $\kappa\colon (S = \mathbb{R}^2) \to M$, $\beta \in \Omega^0(S, \mathfrak{g})$, $\alpha \in \Omega^1(S, \mathfrak{g}^*)$, set

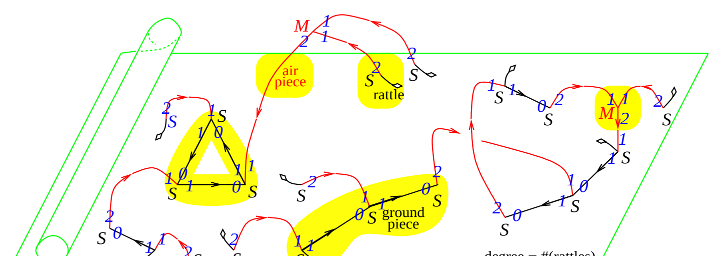
$$\mathcal{O}(A, B, \kappa) := \int \mathcal{D}\beta\mathcal{D}\alpha \exp\left(\frac{i}{\hbar}\int_S \langle \beta, d_{\kappa^*A}\alpha + \kappa^*B\rangle\right).$$


Cattaneo

The BF Feynman Rules. For an edge e, let Φ_e be its direction, in S^3 or S^1. Let ω_3 and ω_1 be volume forms on S^3 and S_1. Then

$$Z_{BF} = \sum_{\substack{\text{diagrams}\\ D}} \frac{[D]}{|\mathrm{Aut}(D)|} \underbrace{\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}}_{S\text{-vertices}} \underbrace{\int_{\mathbb{R}^4}\cdots\int_{\mathbb{R}^4}}_{M\text{-vertices}} \prod_{\substack{\text{red}\\ e\in D}} \Phi_e^*\omega_3 \prod_{\substack{\text{black}\\ e\in D}} \Phi_e^*\omega_1$$

(modulo some STU- and IHX-like relations).



Issues. • Signs don't quite work out, and BF seems to reproduce only "half" of the wheels invariant.
• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
• I don't know how to define "finite type" for arbitrary 2-knots.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-II*, ωεβ/WKO1, ωεβ/WKO2, arXiv:1405.1956, arXiv:1405.1955.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[CT] D. Cimasoni and V. Turaev, *A Lagrangian Representation of Tangles*, Topology **44** (2005) 747–767, arXiv:math.GT/0406269.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links*, Comm. Cont. Math. **3** (2001) 87–136, arXiv:math/9806035.

[LD] J. Y. Le Dimet, *Enlacements d'Intervalles et Représentation de Gassner*, Comment. Math. Helv. **67** (1992) 306–315.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-1405/