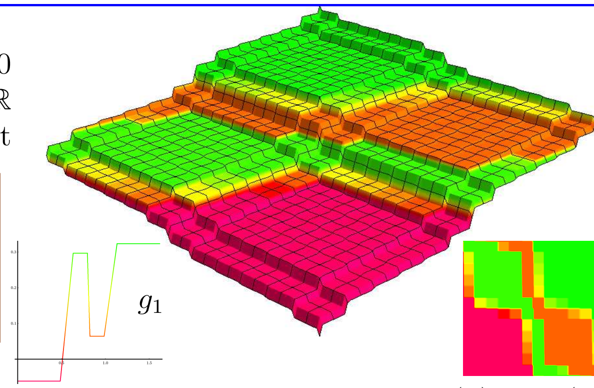
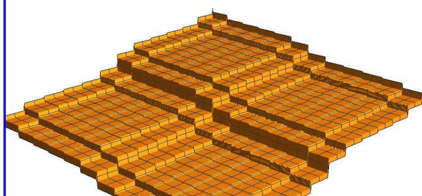
Dessert: Hilbert's 13th Problem, in Full Colour (Page 2)

Step 2. There exists $\phi : [0,1] \to [0,1]$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that $|f(x,y) - g(\phi(x) + \lambda\phi(y))| < \epsilon$ on a set of area at least $1 - \epsilon$ in $[0,1]\times[0,1]$.

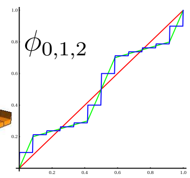
The key. "Iterated poorification".

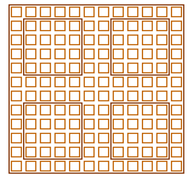


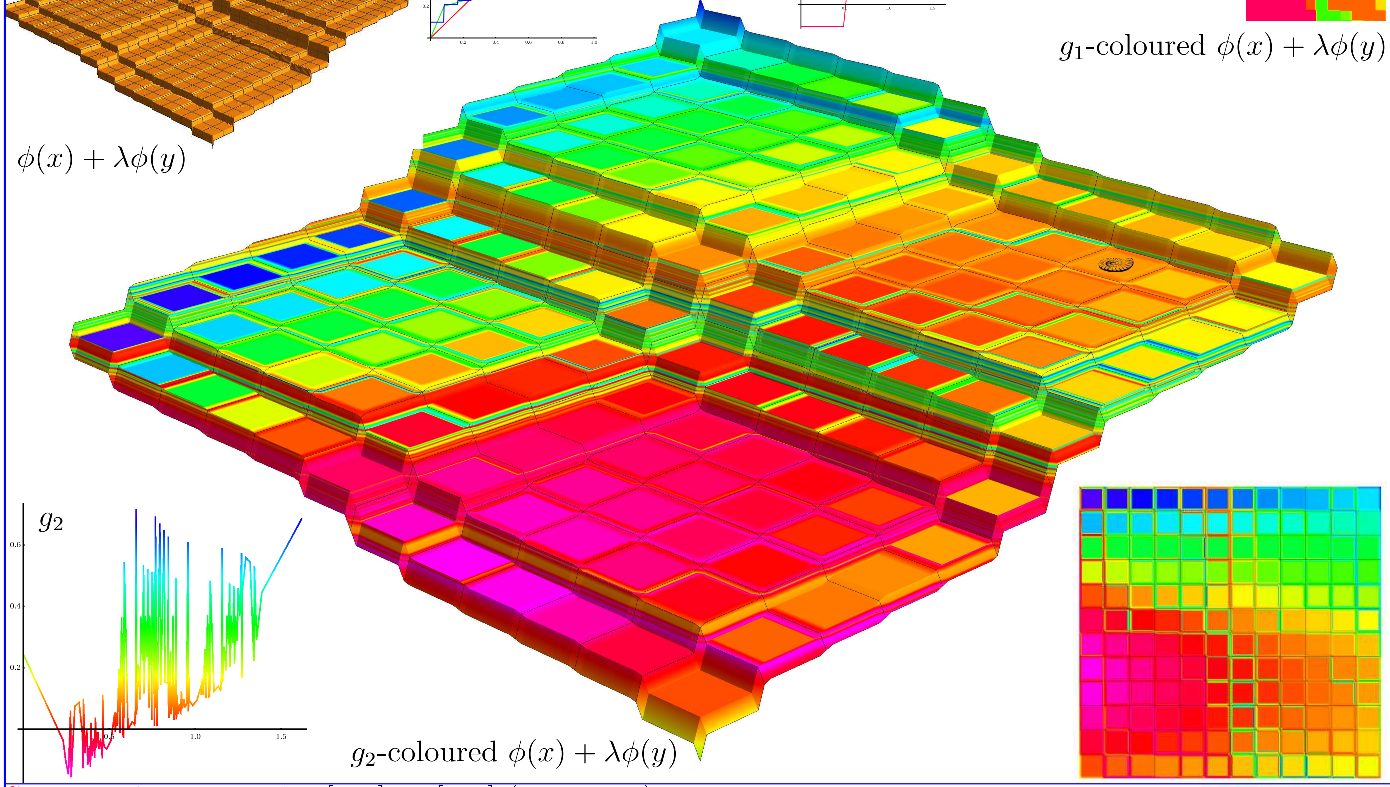
g_1-coloured $\phi(x) + \lambda\phi(y)$



$\phi(x) + \lambda\phi(y)$







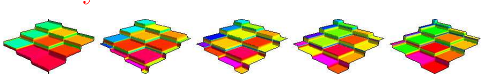
g_2-coloured $\phi(x) + \lambda\phi(y)$

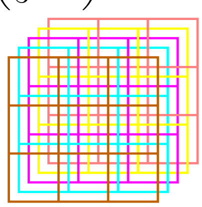
Step 3. There exist $\phi_i : [0,1] \to [0,1]$ $(1 \le i \le 5)$ so that for every $\epsilon > 0$ and every $f : [0,1]\times[0,1] \to \mathbb{R}$ there exists a $g : [0, 1+\lambda] \to \mathbb{R}$ so that

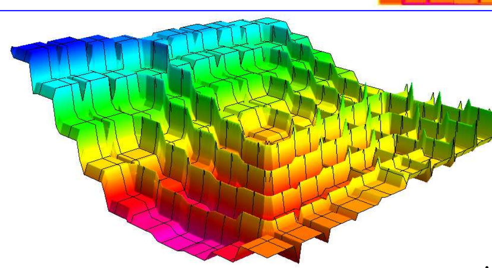
$$|f(x,y) - \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))| < \left(\frac{2}{3} + \epsilon\right) \|f\|_\infty$$

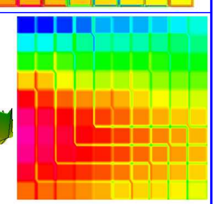
for every $x, y \in [0,1]$.

The key. "Shift the chocolates"...









...then iterate.

Step 4. We are done.

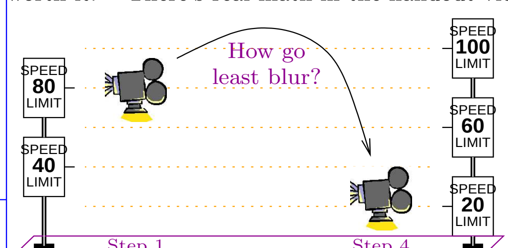
The key. Learn from the artillery!

Set $Tg := \sum_{i=1}^{5} g(\phi_i(x) + \lambda\phi_i(y))$, $f_1 := f$, $M := \|f\|$, and iterate "shooting and adjusting". Find g_1 with $\|g_1\| \le M$ and $\|f_2 := f_1 - Tg_1\| \le \frac{3}{4}M$. Find g_2 with $\|g_2\| \le \frac{3}{4}M$ and $\|f_3 := f_2 - Tg_2\| \le (\frac{3}{4})^2 M$. Find g_3 with $\|g_3\| \le (\frac{3}{4})^2 M$ and $\|f_4 := f_3 - Tg_3\| \le (\frac{3}{4})^3 M$. Continue to eternity. When done, set $g = \sum g_k$ and note that $f = Tg$ as required.

Exercise 1. Do the m-dimensional case.

Exercise 2. Do $\mathbb{R}^m$ instead of just I^m.

Propaganda. I love handouts! • I have nothing to hide and you can take what you want, forwards, backwards, here and at home. • What doesn't fit on one sheet can't be done in one hour. • It takes learning and many hours and a few pennies. The audience's worth it! • There's real math in the handout viewer!



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Fields-1411/