Tangles, Wheels, Balloons

Abstract. I will describe a computable, non-commutative invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

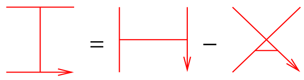
Why I like "non-commutative"? With $FA(x_i)$ the free associative non-commutative algebra,

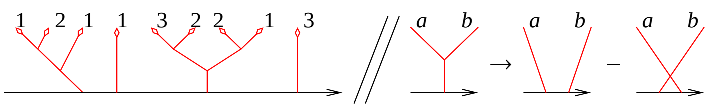
$$\dim \mathbb{Q}[x,y]_d \sim d \ll 2^d \sim \dim FA(x,y)_d.$$

Why I like "computable"?

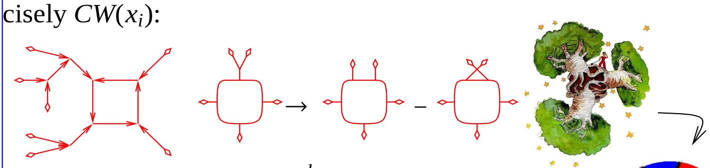
- Because I'm weird.
- Note that π_1 isn't computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i); $FL(x_i)$ = (binary trees with AS vertices and coloured leafs)/(IHX relations). There an obvious map $FA(FL(x_i)) \to FA(x_i)$ defined by $[a,b] \to ab - ba$, which in itself, is IHX.

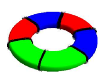




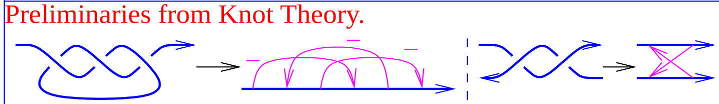
$CW(x_i)$ denotes the vector space of cyclic words in (x_i): $CW(x_i) = FA(x_i)/(x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \to CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1,\dots,n\}$, modulo AS and IHX, is precisely $CW(x_i)$:



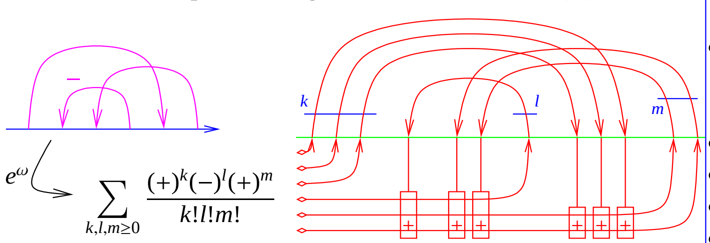
Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

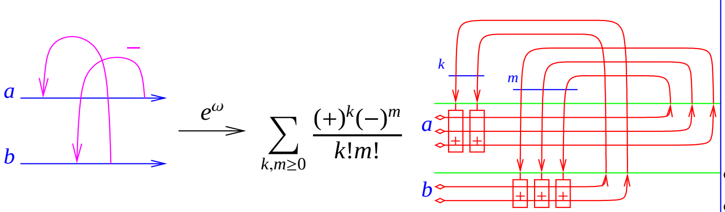


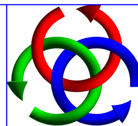
Preliminaries from Knot Theory.



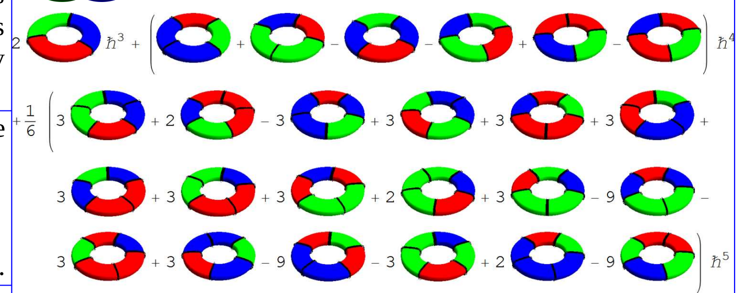
Theorem. ω, the connected part of the procedure below, is an invariant of S-component tangles with values in $CW(S)$:





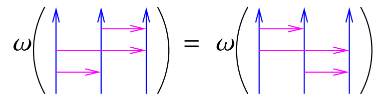


ω is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

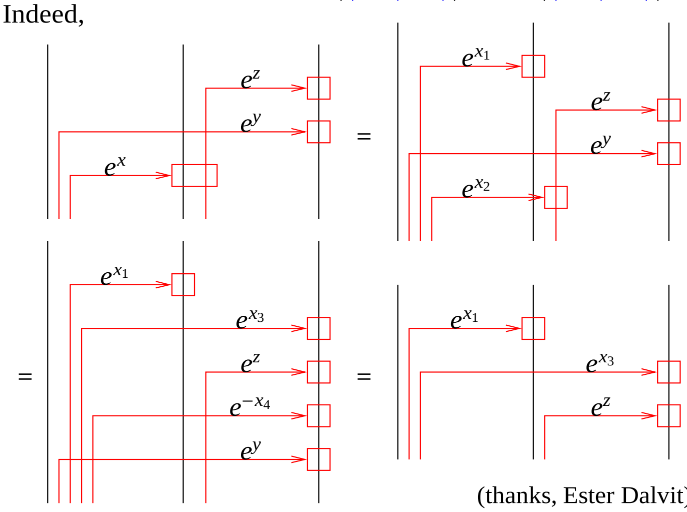


Proof of Invariance.

Need to show:



Indeed,



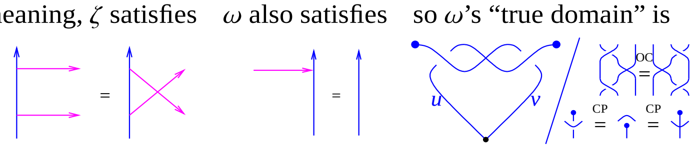
(thanks, Ester Dalvit)

Further Facts

- ω is really the second part of a (trees,wheels)-valued invariant $\zeta = (\lambda, \omega)$. The tree part λ is just a repackaging of the Milnor μ-invariants.
- On u-tangles, ζ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, ω is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine's factorization thereof [Le]) is contained in the Abelianization of ζ [BNS].
- ω vanishes on braids.
- Related to / extends Farber's [Fa]?
- Should be summed and categorified.
- Extends to v and descends to w:

Does ω extend to balloons?

meaning, ζ satisfies ω also satisfies so ω's "true domain" is



- Agrees with BN-Dancso [BND1, BND2] and with [BN].
- ζ, ω are universal finite type invariants.
- Using $\mathcal{X}\!\!\mathcal{K}$: $v\mathcal{K}_n \to w\mathcal{K}_{n+1}$, defines a strong invariant of v-tangles / long v-knots. ($\mathcal{X}\!\!\mathcal{K}$ in LaTeX: ωεβ/zhe)

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Hamilton-1412/