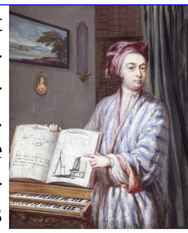
When does a group have a Taylor expansion?

Abstract. It is insufficiently well known that the good old Taylor expansion has a completely algebraic characterization, which generalizes to arbitrary groups (and even far beyond). Thus one may ask: Does the braid group have a Taylor expansion? (Yes, using iterated integrals and/or associators). Do braids on a torus ("elliptic braids") have Taylor expansions? (Yes, using more sophisticated iterated integrals / associators). Do virtual braids have Taylor expansions? (No, yet for nearby objects the deep answer is Probably Yes). Do groups of flying rings (braid groups one dimension up) have Taylor expansions? (Yes, easily, yet the link to TQFT is yet to be fully explored).



Brook Taylor

Disclaimer. I'm asked to talk in a meeting on "iterated integrals", and that's my best. Many of you may think it all trivial. Sorry.

Expansions for Groups. Let G be a group, $\mathcal{K} = \mathbb{Q}G = \{\sum a_i g_i \colon a_i \in \mathbb{Q},\ g_i \in G\}$ its group-ring, $\mathcal{I} = \{\sum a_i g_i \colon \sum a_i = 0\}$ its augmentation ideal. Let

$$\mathcal{A} = \operatorname{gr}\mathcal{K} := \widehat{\bigoplus}_{m\geq 0} \mathcal{I}^m/\mathcal{I}^{m+1}.$$

P.S. $(\mathcal{K}/\mathcal{I}^{m+1})^*$ is Vassiliev / finite-type / polynomial invariants.

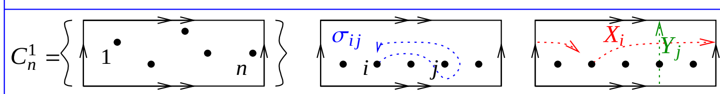
Note that $\mathcal{A}$ inherits a product from G.

Definition. A linear $Z\colon \mathcal{K} \to \mathcal{A}$ is an "expansion" if for any $\gamma \in \mathcal{I}^m$, $Z(\gamma) = (0,\dots,0,\gamma/\mathcal{I}^{m+1},*,\dots)$, a "multiplicative expansion" if in addition it preserves the product, and a "Taylor expansion" if it also preserves the co-product, induced from the diagonal map $G \to G\times G$.

Malcev Quillen

Example. Let $\mathcal{K} = C^\infty(\mathbb{R}^n)$ and $\mathcal{I} = \{f\colon f(0) = 0\}$. Then $\mathcal{I}^m = \{f\colon f \text{ vanishes like } |x|^m\}$ so $\mathcal{I}^m/\mathcal{I}^{m+1}$ degree m homogeneous polynomials and $\mathcal{A} = \{\text{power series}\}$. The Taylor series is the unique Taylor expansion!

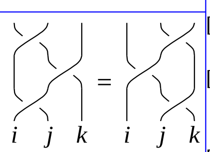
Comment. Unlike lower central series constructions, this generalizes effortlessly to arbitrary algebraic structures.



Elliptic Braids. $PB_n^1 := \pi_1(C_n^1)$ is generated by σ_{ij}, X_i, Y_j, with PB_n relations and $(X_i,X_j) = 1 = (Y_i,Y_j)$, $(X_i,Y_j) = \sigma_{ij}^{-1}$, $(X_iX_j,\sigma_{ij}) = 1 = (Y_iY_j,\sigma_{ij})$, and $\prod X_i$ and $\prod Y_j$ are central. [Bez] implies $\mathcal{A}(PB_n^1) = \langle x_i, y_j\rangle \big/ \big([x_i,x_j] = [y_i,y_j] = [x_i+x_j,[x_i,y_j]] = [y_i+y_j,[x_i,y_j]] = [x_i,\sum y_j] = [y_j,\sum x_i] = 0,\ [x_i,y_j] = [x_j,y_i]\big)$, and [CEE] construct a Taylor expansion using *sophisticated* iterated integrals. [En2] relates this to *Elliptic Associators*.

Virtual Braids. PvB_n is given by the "braids for dummies" presentation:

$$\left\langle \sigma_{ij} \mid \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij},\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij}\right\rangle$$



(every quantum invariant extends to PvB_n!).

By [Lee], $\mathcal{A}(PvB_n)$ is

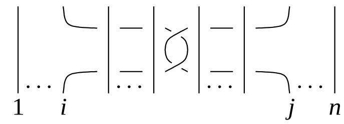
$$\left\langle a_{ij} \mid [a_{ij},a_{ik}] + [a_{ij},a_{jk}] + [a_{ik},a_{jk}] = 0 = [a_{ij},a_{kl}]\right\rangle$$

Theorem [Lee]. While quadratic, PvB_n does not have a Taylor expansion.

Peter Lee

Comment. By the tough theory of quantization of solutions of the classical Young-Baxter equation [EK, En1], PvT_n does have a Taylor expansion. But PvT_n is not a group.

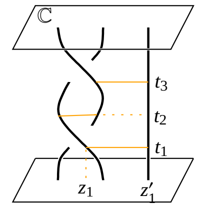
Pure Braids. Take $G = PB_n = \pi_1(C_n = \mathbb{C}^n\backslash\text{diags})$. It is generated by the love-behind-the-bars braids σ_{ij}, modulo "Reidemeister moves". $\mathcal{I}$ is generated by $\{\sigma_{ij} - 1\}$ and $\mathcal{A}$ by $\{t_{ij}\}$, the classes of the $\sigma_{ij} - 1$ in $\mathcal{A}_1 = \mathcal{I}/\mathcal{I}^2$. Reidemeister becomes $[t_{ij} + t_{ik}, t_{jk}] = 0$ and $[t_{ij}, t_{kl}] = 0$.



Theorem. For $\gamma\colon [0,1] \to C_n$, with z_i its ith coordinate, the iterated integral formula on the right defines a Taylor expansion for PB_n.

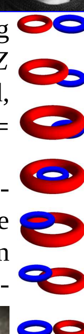
$$Z(\gamma) = \sum_{\substack{m\geq 0\\ 0<t_1<\ldots<t_m<1\\ 1\leq i_1<j_1, i_2<j_2,\ldots,i_m<j_m\leq n}} \!\!\!\!\!\!\!\!\int\quad \prod_{\alpha=1}^m \frac{t_{i_\alpha j_\alpha}}{2\pi i} d\log(z_{i_\alpha} - z_{j_\alpha}),$$

Comments. • I don't know a combinatorial/algebraic proof that PB_n has a Taylor expansion. • Generic "partial expansion" do not extend! • This is the seed for the Drinfel'd theory of associators! • Confession: I don't know a clean derivation of a presentation of PB_n.

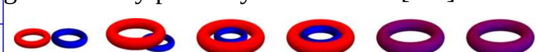


Knizhnik Zamolodchikov Kohno Drinfel'd Kontsevich

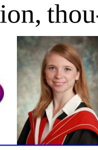
Flying Rings. $PwB_n = PvB_n/(\sigma_{ij}\sigma_{ik} = \sigma_{ik}\sigma_{ij})$ is π_1(flying rings in $\mathbb{R}^3$). $\mathcal{A}(PwB_n) = \mathcal{A}(PvB_n)/[a_{ij},a_{ik}] = 0$, and Z is as easy as it gets: $Z(\sigma_{ij}) = e^{a_{ij}}$ [BP, BND]. Indeed, $Z(\sigma_{ij}\sigma_{ik}\sigma_{jk}) = e^{a_{ij}}e^{a_{ik}}e^{a_{jk}} = e^{a_{ij}+a_{ik}}e^{a_{jk}} = e^{a_{ij}+a_{ik}+a_{jk}} = Z(\sigma_{jk}\sigma_{ik}\sigma_{ij})$.



Comments. • Extends to PwT and generalizes the Alexander polynomial, and even to $PwTT$ and interprets the Kashiwara-Vergne problem [BND]. • I don't know an iterated-integral derivation, or any TQFT derivation, though BF theory probably comes close [CR].



The "Vertex" in TT.



Dancso

References.

Paper in Progress: ωεβ/ExQu

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: Braids, Knots and the Alexander Polynomial*, ωεβ/WKO1, arXiv:1405.1956; and *II: Tangles and the Kashiwara-Vergne Problem*, ωεβ/WKO2, arXiv:1405.1955.

[BP] B. Berceanu and S. Papadima, *Universal Representations of Braid and Braid-Permutation Groups*, J. of Knot Theory and its Ramifications **18-7** (2009) 973–983, arXiv:0708.0634.

[Bez] R. Bezrukavnikov, *Koszul DG-Algebras Arising from Configuration Spaces*, Geom. Func. Anal. **4-2** (1994) 119–135.

[CEE] D. Calaque, B. Enriquez, and P. Etingof, *Universal KZB Equations I: The Elliptic Case*, Prog. in Math. **269** (2009) 165–266, arXiv:math/0702670.

[CR] A. S. Cattaneo and C. A. Rossi, *Wilson Surfaces and Higher Dimensional Knot Invariants*, Commun. in Math. Phys. **256-3** (2005) 513–537, arXiv:math-ph/0210037.

[En1] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras*, Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[En2] B. Enriquez, *Elliptic Associators*, Selecta Mathematica **20** (2014) 491–584, arXiv:1003.1012.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I*, Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[Lee] P. Lee, *The Pure Virtual Braid Group Is Quadratic*, Selecta Mathematica **19-2** (2013) 461–508, arXiv:1110.2356.