Commutators

Abstract. The commutator of two elements x and y in a group G is $xyx^{-1}y^{-1}$. That is, x followed by y followed by the inverse of x followed by the inverse of y. In my talk I will tell you how commutators are related to the following four riddles:

1. Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?
2. Can you hang a picture on a string on the wall using n nails, so that if you remove any one of them, the picture will fall?
3. Can you draw an n-component link (a knot made of n non-intersecting circles) so that if you remove any one of those n components, the remaining $(n-1)$ will fall apart?
4. Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator?

Definition. The commutator of two elements x and y in a group G is $[x,y] := xyx^{-1}y^{-1}$.

Example 1. In S_3, $[(12),(23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$,

$$[(ij),(jk)] = (ijk).$$

Example 2. In $S_{\geq 4}$,

$$[(ijk),(jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk).$$

Example 3. In $S_{\geq 5}$,

$$[(ijk),(klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm).$$

Example 4. So, in fact, in S_5, $(123) = [(412),(253)] = [[(341),(152)],[(125),(543)]] = [[[(234),(451)],[(315),(542)]],[[(312),(245)],[(154),(423)]]] = [\ [[[(123),(354)],[(245),(531)]],[[(231),(145)],[(154),(432)]]], [[[(431),(152)],[(124),(435)]],[[(215),(534)],[(142),(253)]]]\ ]$.

Solving the Quadratic, $ax^2+bx+c=0$: $\delta = \sqrt{\Delta}$; $\Delta = b^2-4ac$; $r = \frac{\delta-b}{2a}$.

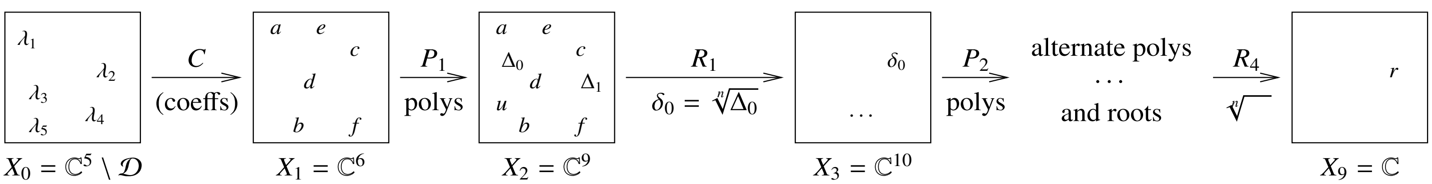
Solving the Cubic, $ax^3+bx^2+cx+d=0$: $\Delta = 27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2$; $\delta = \sqrt{\Delta}$; $\Gamma = 27a^2d-9abc+3\sqrt{3}a\delta+2b^3$; $\gamma = \sqrt[3]{\frac{\Gamma}{2}}$; $r = -\frac{\frac{b^2-3ac}{\gamma}+b+\gamma}{3a}$.

Solving the Quartic, $ax^4+bx^3+cx^2+dx+e=0$: $\Delta_0 = 12ae-3bd+c^2$; $\Delta_1 = -72ace+27ad^2+27b^2e-9bcd+2c^3$; $\Delta_2 = \frac{1}{27}\left(\Delta_1^2-4\Delta_0^3\right)$; $u = \frac{8ac-3b^2}{8a^2}$; $v = \frac{8a^2d-4abc+b^3}{8a^3}$; $\delta_2 = \sqrt{\Delta_2}$; $Q = \frac{1}{2}\left(3\sqrt{3}\delta_2+\Delta_1\right)$; $q = \sqrt[3]{Q}$; $S = \frac{\frac{\Delta_0}{q}+q}{12a} - \frac{u}{6}$; $s = \sqrt{S}$; $\Gamma = -\frac{v}{s}-4S-2u$; $\gamma = \sqrt{\Gamma}$; $r = -\frac{b}{4a}+\frac{\gamma}{2}+s$.

Theorem. The is no general formula, using only the basic arithmetic operations and taking roots, for the solution of the quintic equation $ax^5+bx^4+cx^3+dx^2+ex+f=0$.

Key Point. The "persistent root" of a closed path (path lift, in topological language) may not be closed, yet the persistent root of a commutators of closed paths is always closed.

Proof. Suppose there was a formula, and consider the corresponding "composition of machines" picture:



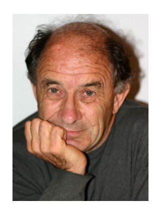
Now if $\gamma_1^{(1)},\gamma_2^{(1)},\ldots,\gamma_{16}^{(1)}$, are paths in X_0 that induce permutations of the roots and we set $\gamma_1^{(2)} := [\gamma_1^{(1)},\gamma_2^{(1)}]$, $\gamma_2^{(2)} := [\gamma_3^{(1)},\gamma_4^{(1)}]$, …, $\gamma_8^{(2)} := [\gamma_{15}^{(1)},\gamma_{16}^{(1)}]$, $\gamma_1^{(3)} := [\gamma_1^{(2)},\gamma_2^{(2)}]$, …, $\gamma_4^{(3)} := [\gamma_7^{(2)},\gamma_8^{(2)}]$, $\gamma_1^{(4)} := [\gamma_1^{(3)},\gamma_2^{(3)}]$, $\gamma_2^{(4)} := [\gamma_3^{(3)},\gamma_4^{(3)}]$, and finally $\gamma^{(5)} := [\gamma_1^{(4)},\gamma_2^{(4)}]$ (all of those, commutators of "long paths"; I don't know the word "homotopy"), then $\gamma^{(5)}/\!\!/C/\!\!/P_1/\!\!/R_1/\!\!/\cdots/\!\!/R_4$ is a closed path. Indeed,

- In X_0, none of the paths is necessarily closed.
- After C, all of the paths are closed.
- After P_1, all of the paths are still closed.
- After R_1, the $\gamma^{(1)}$'s may open up, but the $\gamma^{(2)}$'s remain closed.

…

- At the end, after R_4, $\gamma^{(4)}$'s may open up, but $\gamma^{(5)}$ remains closed.

But if the paths are chosen as in Example 4, $\gamma^{(5)}/\!\!/C/\!\!/P_1/\!\!/R_1/\!\!/\cdots/\!\!/R_4$ is not a closed path. □


V.I. Arnold

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/CMU-1504/ and at http://www.math.toronto.edu/~drorbn/Talks/Sydney-1708