Dror Bar-Natan: Talks: LesDiablerets-1508:
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/

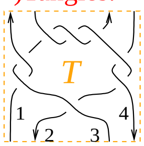
Work in Progress on

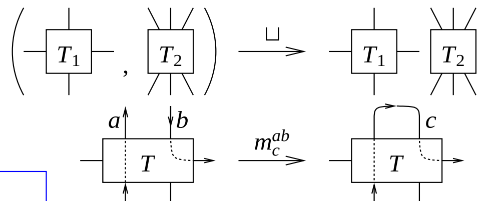
Polynomial Time Knot Polynomials, A

Abstract. The value of things is inversely correlated with their computational complexity. "Real time" machines, such as our brains, only run linear time algorithms, and there's still a lot we don't know. Anything we learn about things doable in linear time is truly valuable. Polynomial time we can in-practice run, even if we have to wait; these things are still valuable. Exponential time we can play with, but just a little, and exponential things must be beautiful or philosophically compelling to deserve attention. Values further diminish and the aesthetic-or-philosophical bar further rises as we go further slower, or un-computable, or ZFC-style intrinsically infinite, or large-cardinalish, or beyond.

I will explain some things I know about polynomial time knot polynomials and explain where there's more, within reach.

(v-)Tangles.



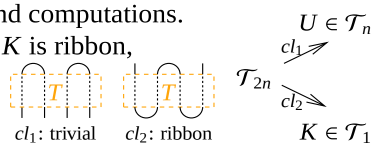


(meta-associativity: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$)

Why Tangles?

- Finitely presented.
- Divide and conquer proofs and computations.
- "Algebraic Knot Theory": If K is ribbon, $z(K) \in \{cl_2(\zeta)\colon cl_1(\zeta) = 1\}$.

(Genus and crossing number are also definable properties).



A blackboard aside on genus?

Faster is better, leaner is meaner!

Theorem 1.

$\exists!$ an invariant z_0: {pure framed S-component tangles} $\to \Gamma_0(S) := R \times M_{S\times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c}\omega_1 & S_1\\ \hline S_1 & A_1\end{array}, \begin{array}{c|c}\omega_2 & S_2\\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a, T_b \to T_c,\ \mu := 1-\beta]{m^{ab}_c} \left(\begin{array}{c|cc}\mu\omega & c & S\\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu\\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu\end{array}\right),$$

and satisfying $\left(\uparrow_a;\ {}_a{\nwarrow\!\!\nearrow}_b, {}_b{\nwarrow\!\!\nearrow}_a\right) \xrightarrow{z_0} \left(\begin{array}{c|c}1 & a\\ \hline a & 1\end{array};\ \begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array}\right).$

In Addition

- The matrix part is just a stitching formula for Burau/Gassner [LD, KLW, CT].
- $K \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A-I)/(1-T')$ is the MVA, mod units.
- The fastest Alexander algorithm I know.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.

M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Implementation

key idea: $(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

ωεβ/Demo

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
  M = Outer[Factor[∂h_#1 t_#2 λ] &, S, S];
  M = Prepend[M, t# & /@ S] // Transpose;
  M = Prepend[M, Prepend[h# & /@ S, ω]];
  M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ma_ b_→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ; ∂tb,ha λ  ∂tb,hb λ  ∂tb λ; ∂ha λ  ∂hb λ  λ) /. (t | h)a|b → 0;
  Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{hc, 1}]
   /. {Ta → Tc, Tb → Tc} // ΓCollect];
Rpa_ b_ := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rma_ b_ := Rpa b /. Ta → 1 / Ta;
```

Meta-Associativity

Runs.

```
ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
```

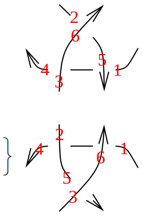
```
True
```

R3

. . . divide and conquer!

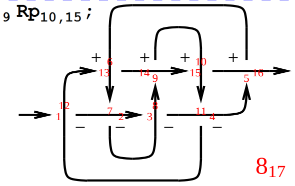
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$




```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}

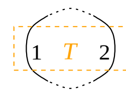
Closed Components.

The Halacheva trace tr_c satisfies $m^{ab}_c /\!/ \mathrm{tr}_c = m^{ba}_c /\!/ \mathrm{tr}_c$ and computes the MVA for all links in the atlas, but its domain is not understood:

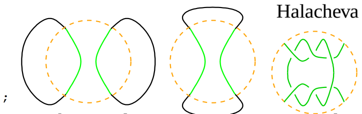
$$\begin{array}{c|cc}\omega & c & S\\ \hline c & \alpha & \theta\\ S & \psi & \Xi\end{array} \xrightarrow[\mu := 1-\alpha]{\mathrm{tr}_c} \begin{array}{c|c}\mu\omega & S\\ \hline S & \Xi+\psi\theta/\mu\end{array}$$

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂tc,hc λ  ∂tc λ; ∂hc λ  λ) /. (t | h)c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

```
True
```



Halacheva



cl_1: trivial cl_2: ribbon example

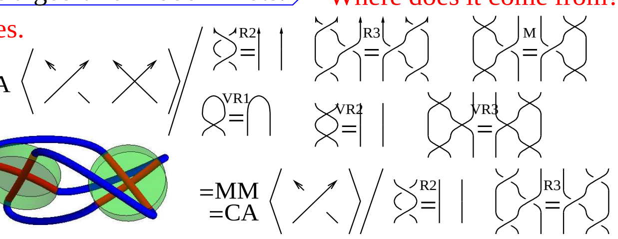
Weaknesses.

- m^{ab}_c and tr_c are non-linear.
- The product ωA is always Laurent, but my current proof takes induction with exponentially many conditions.
- I still don't understand tr_c, "unitarity", the algebra for ribbon knots.

Where does it come from?

v-Tangles.

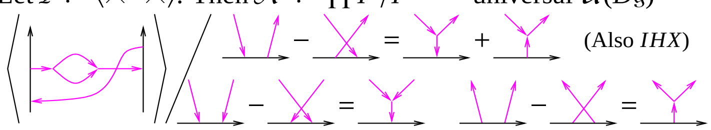
$v\mathcal{T}$:=PA



flying pogs

=MM
=CA

Let $\mathcal{I} := \langle \times - \times \rangle$. Then $\mathcal{A}^v := \prod I^n/I^{n+1}$ ="universal $\mathcal{U}(D\mathfrak{g})^{\otimes S}$"=



(Also *IHX*)

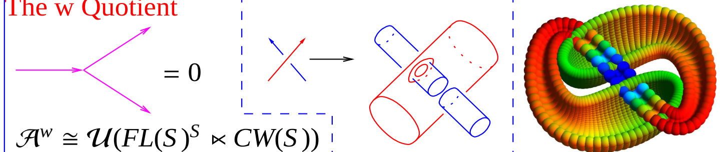
Fine print: No sources no sinks, AS vertices, internally acyclic, deg = (#vertices)/2.

Likely Theorem.

[EK, En] There exists a homomorphic expansion (universal finite type invariant) $Z\colon v\mathcal{T} \to \mathcal{A}^v$. (issues suppressed)

Too hard! Let's look for "meta-monoid" quotients.

The w Quotient



$\mathcal{A}^w \cong \mathcal{U}(FL(S)^S \ltimes CW(S))$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/