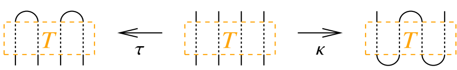
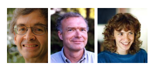
Work in Progress!

Gauss-Gassner Invariants, What?

Abstract. In a "degree d Gauss diagram formula" one produces a number by summing over all possibilities of paying very close attention to d crossings in some n-crossing knot diagram while observing the rest of the diagram only very loosely, minding only its skeleton. The result is always poly-time computable as only $\binom{n}{d}$ states need to be considered. An under-explained paper by Goussarov, Polyak, and Viro [GPV] shows that every type d knot invariant has a formula of this kind. Yet only finitely many integer invariants can be computed in this manner within any specific polynomial time bound.
I suggest to do the same as [GPV], except replacing "the skeleton" with "the Gassner invariant", which is still poly-time. One poly-time invariant that arises in this way is the Alexander polynomial (in itself it is infinitely many numerical invariants) and I believe (and have evidence to support my belief) that there are more.

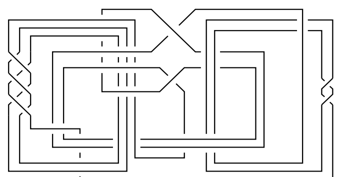
The QUILT Target. QUick Invariants of Large Tangles, for little had been found since Alexander (and if they're there, how can we not know all about them?), and for {ribbon} ≠ {slice}:





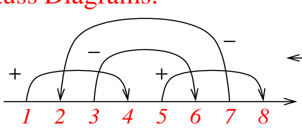
Gompf, Scharlemann, Thompson [GST]

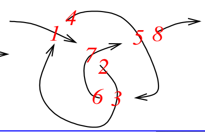
$$\begin{array}{ccc} U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\ \nearrow\tau & & \nearrow\tau \\ \mathcal{T}_{2n} & \xrightarrow{z} & \mathcal{A}_{2n} \\ \searrow\kappa & & \searrow\kappa \\ \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{A}_1 \end{array}$$



Faster is better, leaner is meaner!

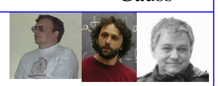
Gauss Diagrams. (just QUILK, today)



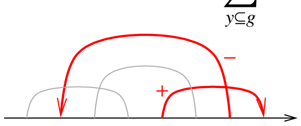


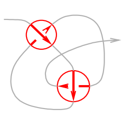
Gauss

Gauss Diagram Formulas [PV, GPV]. If g is a Gauss diagram and F an unsigned Gauss diagram, $\langle F,g\rangle_{PV} := \sum_{y\subseteq g}(-1)^y\delta(F,\bar{y})$:

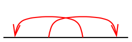


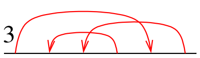
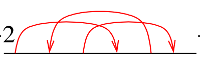
Goussarov-Polyak-Viro





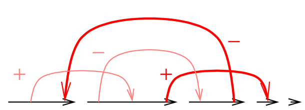
Under-Explain Theorem [GPV]. Every finite type invariant arises in this way.

$F_2 =$  $\Rightarrow \langle F_2,K\rangle = v_2(K)$

$F_3 =$ 3  +2  + rotations

$\Rightarrow \langle F_3,K\rangle = 6v_3(K)$

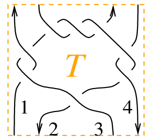
Gauss-Gassner Invariants. Want more? Increase your environmental awareness! Instead of nearly-forgetting y^c, compute its Burau/Gassner invariant (note that y^c is a tangle in a Swiss cheese; more easily, a virtual tangle):

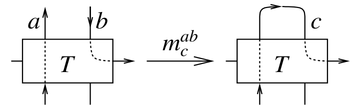


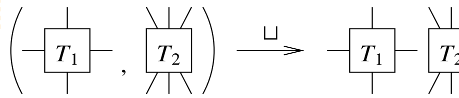
$$GG_{k,F}(g) = \sum_{y\subseteq g,\,|y|\leq k}\bar{F}(y,z(y^c)) = \sum_{y\subseteq g,\,|y|\leq k} F(y,z(g\text{ cut near }y)),$$

where k is fixed and $F(y,\gamma)$ is a function of a list of arrows y and a square matrix γ of side $|y|+1\leq k+1$.

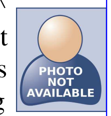
The (Burau-)Gassner Invariant.







Burau



Gassner

Theorem 1. ∃! an invariant z: {pure framed S-component tangles} → $\Gamma(S) := M_{S\times S}(R_S)$, where $R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c} & S_1 \\ \hline S_1 & A_1\end{array},\ \begin{array}{c|c} & S_2 \\ \hline S_2 & A_2\end{array}\right) \xrightarrow{\sqcup} \begin{array}{c|cc} & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array},$$

$$\begin{array}{c|ccc} & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\substack{T_a,T_b\to T_c \\ \mu:=1-\beta}]{m^{ab}_c} \left(\begin{array}{c|cc} & c & S \\ \hline c & \gamma+\alpha\delta/\mu & \epsilon+\delta\theta/\mu \\ S & \phi+\alpha\psi/\mu & \Xi+\psi\theta/\mu \end{array}\right),$$

and satisfying $\left(|_a;\ {}_a\!\nearrow\!\!\nwarrow_b,\ {}_b\!\nearrow\!\!\nwarrow_a\right) \xrightarrow{z} \left(\begin{array}{c|c} & a \\ \hline a & 1\end{array};\ \begin{array}{c|cc} & a & b \\ \hline a & 1 & 1-T_a^{\pm1} \\ b & 0 & T_a^{\pm1}\end{array}\right)$.

See also [LD, KLW, CT, BNS].

Theorem 2. With $k=1$ and F_A defined by

$$F_A(\xrightarrow{s},\gamma) = s\left.\frac{\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{32}}{\gamma_{33}+\gamma_{13}\gamma_{32}-\gamma_{12}\gamma_{33}}\right|_{T_a\to T},$$

$$F_A(\xleftarrow{s},\gamma) = s\left.\frac{\gamma_{13}\gamma_{32}-\gamma_{12}\gamma_{33}}{\gamma_{32}-\gamma_{23}\gamma_{32}+\gamma_{22}\gamma_{33}}\right|_{T_a\to T},$$

$GG_{1,F_A}(K)$ is a regular isotopy invariant. Unfortunately, for every knot K, $GG_{1,F_A}(K) - T\frac{d}{dT}\log A(K)(T) \in \mathbb{Z}$, where $A(K)$ is the Alexander polynomial of K.

Expectation. Higher Gauss-Gassner invariants exist . . .
(though right now I can reach for them only wearing my exoskeleton)

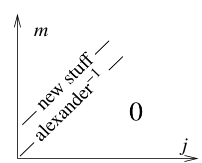
Jones, Melvin, Morton, Rozansky, Garoufalidis

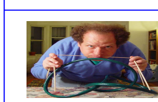
. . . and they are the "higher diagonals" in the MMR expansion of the coloured Jones polynomial J_λ.

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar) = 1$.





Help Needed!

I'm slow and feeble-minded.