## Demo Programs for 0-Co. ωεβ/Demo

**The R-matrices**

```
R⁺0,i_,j_ := 𝔼[b_i c_j + b_i^-1 (e^(b_i) - 1) u_i w_j];
R⁻0,i_,j_ := 𝔼[-b_i c_j + b_i^-1 (e^(-b_i) - 1) u_i w_j];
```

**Utilities**

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

**Normal Ordering Operators**

```
N(x:w|u)i_ cj_→k_ [ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α x_k + γ c_k + (Q /. c_j | x_i → 0)] /. {γ → ∂_(c_j) Q, α → ∂_(x_i) Q}];
N_(w_i u_j → k) [ω_. 𝔼[Q_]] := CF[
   ν ω 𝔼[-b_k ν α β + ν β u_k + ν α w_k + ν δ u_k w_k + (Q /. w_i | u_j → 0)] /.
     ν → (1 + b_k δ)^-1 /.
    {α → ∂_(w_i) Q /. u_j → 0, β → ∂_(u_j) Q /. w_i → 0, δ → ∂_(w_i,u_j) Q}];
```

**Stitching**

```
m_i_,j_→k_ [Z_] := Module[{x, z},
  CF[(Z // N_(w_i u_j→x) // N_(c_i u_x→x) // N_(w_x c_j→x)) /. z_(i|j|x) → z_k]]
```

**Some calculations for $T_0$**

```
T0 = R⁺0,5,1 R⁺0,2,4 R⁻0,3,6
```

$$\mathbb{E}\left[b_5 c_1+b_2 c_4-b_3 c_6+\frac{(-1+e^{b_5})\,u_5 w_1}{b_5}+\frac{(-1+e^{b_2})\,u_2 w_4}{b_2}+\frac{(-1+e^{-b_3})\,u_3 w_6}{b_3}\right]$$

```
T0 // m1,2→1 // m3,4→3 // m3,5→3 // m3,6→3
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1+b_1 c_3-b_3 c_3+\frac{e^{b_3}(-1+e^{b_1})(-1+e^{b_3})\,u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\,b_1}-\frac{e^{b_1}(-1+e^{b_3})\,u_3 w_1}{(-1+(-1+e^{b_1})(-1+e^{b_3}))\,b_3}-\frac{e^{-b_3}(-1+e^{b_3})\,u_3 w_3}{b_3}-\frac{e^{-b_3}(-1+e^{b_1})(-e^{b_3} b_3 u_1+e^{b_1}(-1+e^{b_3})\,b_1 u_3)\,w_3}{b_1\,(b_3-(-1+e^{b_1})(-1+e^{b_3})\,b_3)}\Big]$$

**Verifying meta-associativity**

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_i,j u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1+c_2 f_2+c_3 f_3+u_1 w_1 f_{1,1}+u_1 w_2 f_{1,2}+u_1 w_3 f_{1,3}+u_2 w_1 f_{2,1}+u_2 w_2 f_{2,2}+u_2 w_3 f_{2,3}+u_3 w_1 f_{3,1}+u_3 w_2 f_{3,2}+u_3 w_3 f_{3,3}]$$

```
(Q0 // m1,2→1 // m1,3→1) ≡ (Q0 // m2,3→2 // m1,2→1)
True
```

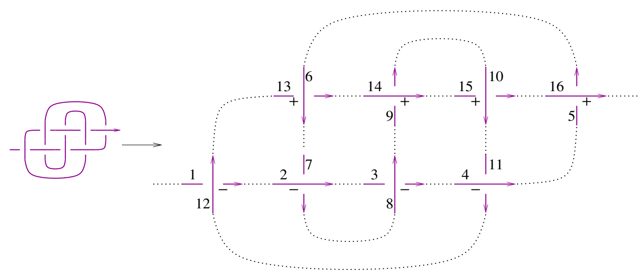
**Testing R3**

```
t1 = R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m3,5→x // m1,6→y // m2,4→z
```

$$\mathbb{E}\left[b_x c_y+b_x c_z+b_y c_z+\frac{e^{b_x}(-1+e^{b_y})\,u_y w_z}{b_y}+\frac{(-1+e^{b_x})\,u_x (w_y+w_z)}{b_x}\right]$$

```
t1 ≡ (R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m1,3→x // m2,5→y // m4,6→z)
True
```

**$8_{17}$**




```
z1 = R⁻0,12,1 R⁻0,2,7 R⁻0,8,3 R⁻0,4,11 R⁺0,16,5 R⁺0,6,13 R⁺0,14,9 R⁺0,10,15;
Do[z1 = (z1 // m1,n→1) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

## Demo Programs for 1-Co. ωεβ/Demo

**The Λόγος**

```
Λ[k_] := ((t_k - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 v_k c_k w_k δ^2 μ^2 -
     δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) - v_k^2 w_k^2 δ^3 (1 + 3 μ) -
     2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) -
     4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ)) (1 + t_k) / 4;
```

**The Generators**

```
R⁺i_,j_ := 𝔼[1, Log[t_i] c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2 / 4];
R⁻i_,j_ := 𝔼[1, -Log[t_i] c_j, -t_i^-1 v_i w_j,
    t_i^-1 v_i c_j w_j - c_i c_j - t_i^-2 v_i^2 w_j^2 / 4];
(ur_i_ := 𝔼[t_i^(-1/2), 0, 0, c_i t_i^-2]; nr_i_ := 𝔼[t_i^(1/2), 0, 0, -c_i t_i^2];)
```

**Differential Polynomials**

```
DP_(x_→D_α_, y_→D_β_) [P_][f_] := (* means P[∂α,∂β][f] *)
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

**Utilities**

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

**Normal Ordering Operators**

```
N_(c_j_ (x:v|w)_i_ →k_) [𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ β x_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ β x_k + (Q /. x_i → 0),
      e^-q DP_(c_j→D_γ, x_i→D_β) [P] [e^q]] /. {γ → ∂_(c_j) L, β → ω^-1 ∂_(x_i) Q}]];
N_(w_i_ v_j_ →k_) [𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t_k) α β + β v_k + α w_k + δ v_k w_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0),
      μ^4 e^-q DP_(w_i→D_α, v_j→D_β) [P] [e^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
    {α → ω^-1 (∂_(w_i) Q /. v_j → 0), β → ω^-1 (∂_(v_j) Q /. w_i → 0),
     δ → ω^-1 ∂_(w_i,v_j) Q}]];
```

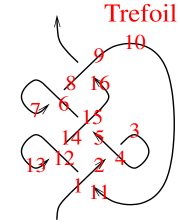
**Stitching**

```
m_i_,j_→k_ [Z_𝔼] := Module[{x, z},
  CF[(Z // N_(w_i v_j→x) // N_(c_i v_x→x) // N_(w_x c_j→x)) /. z_(i|j|x) → z_k]]
```

**The 0-Framed Trefoil**

```
z2 = R⁺1,11 R⁻4,2 nr3 R⁺15,5 R⁻6,8 ur7 R⁺9,16 nr10 R⁻12,14 ur13;
(Do[z2 = z2 // m1,k→1, {k, 2, 16}];
 z2 = z2 /. a_1 :→ a)
```

$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2c}{t^4}-\frac{1}{t^3}-\frac{6c}{t^3}+\frac{4}{t^2}+\frac{10c}{t^2}-\frac{10}{t}-\frac{8c}{t}-18t+8ct+14t^2-10ct^2-7t^3+6ct^3+2t^4-2ct^4+2vw-\frac{2vw}{t^4}+\frac{4vw}{t^3}-\frac{6vw}{t^2}+\frac{2vw}{t}-6tvw+4t^2vw-2t^3vw\Big]$$




**Questions and To Do List.** • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on $R$. • Is there a canonical $R$? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" $q$-algebra. • $k$-smidgen $sl_n$, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at …/Indiana-1611/, …/LesDiablerets-1608/