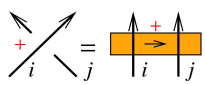
1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b,\cdot] = 0,\ [c, u] = u,\ [c, w] = -w,\ [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c, w\rangle$ and (b, u) is the dual basis of (c, w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $\mathcal{S}(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,
$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 \middle| x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) \middle| i\colon u_i,\ j\colon c_j w_j\right)$

Example. $Z(T_0) = \qquad = \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n.$

$$\mathbb{O}\Big(\exp\left(b_5 c_1 + \tfrac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \tfrac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \tfrac{e^{-b_3}-1}{b_3} u_3 w_6\right) \Big|$$
"ucw form"
$$x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6\Big) = \mathbb{O}\left(\zeta \middle| x\colon u_x c_x w_x,\ y\colon u_y c_y w_y\right)$$

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1a. $N^{cu} := \mathbb{O}(e^{\gamma c + \beta u}|uc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c} = e^{\gamma c}e^{e^{\gamma}\beta u}$

1b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w}|wc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w}|cw)$ …in the $\{ax + b\}$ group)

2. $\mathbb{O}(e^{\alpha w + \beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u}|uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u} = e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw}|wu) = \mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

5. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw}|wu) \stackrel{\rightarrow}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw}|uw)$

6. $N_k^{c_i c_j} := \mathbb{O}(\zeta|c_i c_j) \stackrel{\rightarrow}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k)|c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition
$$u_i c_i \overline{w_i u_j}\, c_j w_j \xrightarrow{N_x^{w_i u_j}} u_i\, \overline{c_i u_x}\, \overline{w_x c_j}\, w_j \xrightarrow{N_x^{c_i u_x} /\!/ N_x^{w_x c_j}} \overline{u_i u_x}\, \overline{c_x c_x}\, \overline{w_x w_j} \xrightarrow{i,j,x \to k} u_k c_k w_k$$

On to 1-smidgen invariants, where much is the same…

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw}|wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda)e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)}|ucw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, u, c, and w:
$$\begin{aligned}\Lambda = &- b\nu(\alpha^2\beta^2\nu^2 + 4\alpha\beta\delta\nu + 2\delta^2)/2 + \beta^2\delta\nu^3(b\delta + 2)u^2/2\\ &+ \delta^3\nu^3(3b\delta + 4)u^2w^2/2 + \beta\delta^2\nu^3(2b\delta + 3)u^2 w\\ &+ \alpha\delta^2\nu^3(2b\delta + 3)uw^2 + 2\delta\nu^2(b\delta + 2)(\alpha\beta\nu + \delta)uw\\ &+ \alpha^2\delta\nu^3(b\delta + 2)w^2/2 + 2(\alpha\beta\nu + \delta)c + 2\beta\delta\nu uc + 2\delta^2\nu ucw\\ &+ 2\alpha\delta\nu cw + \beta\nu^2(\alpha\beta\nu + 2\delta)u + \alpha\nu^2(\alpha\beta\nu + 2\delta)w.\end{aligned}$$

Proof. A lengthy computation. (Verification: ωεβ/Big)

Problem. We now need to normal-order perturbed Gaussians!

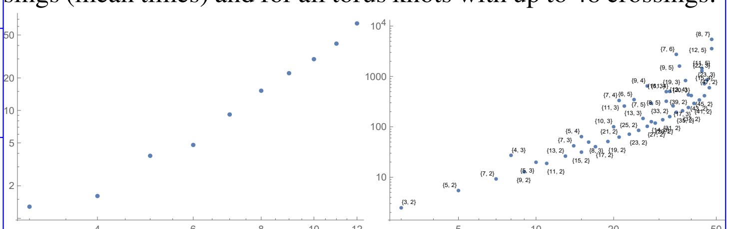
Solution. Borrow some tactics from QFT:
$$\mathbb{O}(\epsilon P(c, u)e^{\gamma c + \beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c + \beta u}|uc) =$$
and likewise $\qquad \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c + e^{-\gamma}\beta u}|cu),$
$$\mathbb{O}\left(\epsilon P(u, w)e^{\alpha w + \beta u + \delta uw}|wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha)\nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)}|ucw\right)$$

Finally, the values of the generators ⤯, ⤰, $\overrightarrow{n}$, and $\underset{\rightarrow}{u}$, are set by solving many equations, non-uniquely.

Pragmatic Simplifications. Set $t := e^b$, work with $v := (t - 1)u/b$, and set $\mathbb{E}(\omega, L, Q, P) := \mathbb{O}\left(\omega^{-1}e^{L + Q/\omega}(1 + \epsilon\omega^{-4}P)\colon (i\colon v_i c_i w_i)\right)$. Now $\omega \in R_S := \mathbb{Z}[t_i, t_i^{-1}]$ is Laurent, $L = \sum l_{ij}\log(t_i)c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij}v_i w_j$ with $q_{ij} \in R_S$, and P is a quartic polynomial in v_i, c_j, w_k with coefficients in R_S. The operations are lightly modified, and the Λόγος and the values of the generators become somewhat simpler, as in the implementation below.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.
$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}\underbrace{n^2}_{G}}^{B} = n^3w^4 \in [n^5, n^7]$$
A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that
$$P = A^2\frac{(t - 1)^3\rho_1 + t^2(2vw + (1 - t)(1 - 2c))AA'}{(1 - t)t}.$$
Furthermore, A and ρ_1 are symmetric under $t \to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t) = \rho_1^+(t) + \rho_1^+(t^{-1}) - \rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always $\deg\rho_1^+ \le 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

This is `http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/`. Better videos at …/Indiana-1611/, …/LesDiablerets-1608/