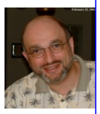
A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Fluid! Help Needed!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.


Rozansky


Overbay

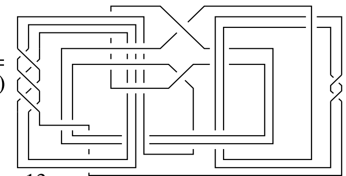

Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$ $\quad 1 \in \mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$

Faster is better, leaner is meaner!



$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

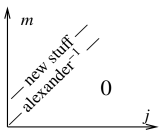
$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing


Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar) = 1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q-q^{-1})A(K)(q^d)}\left(1 + \sum_{k=1}^\infty \frac{(q-1)^k R_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has

$$\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\} \text{ with } R_S := \mathbb{Z}(\{t_a\colon a\in S\}):$$

$$\left(_a\nearrow_b, {}_b\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a,t_b\to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

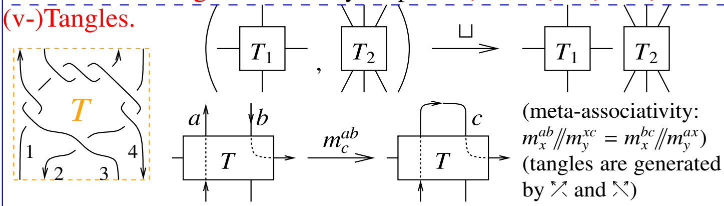
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

Why "spectacular"? Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Also, will bound genus and may disprove {ribbon} = {slice}.

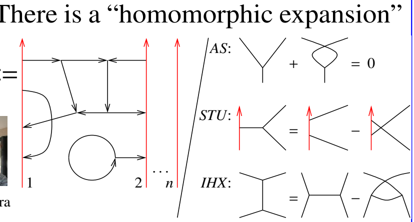
(v-)Tangles.

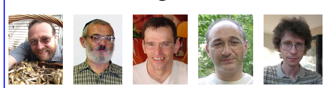


(meta-associativity: $m_x^{ab}/\!\!/m_y^{xc} = m_x^{bc}/\!\!/m_y^{ax}$) (tangles are generated by $\nearrow$ and $\nwarrow$)

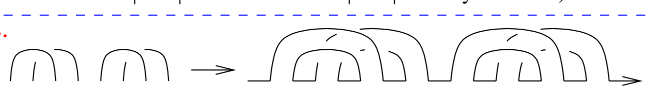
Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

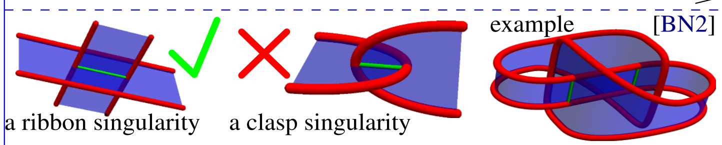
$$Z\colon \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}_S^v :=$$




Etingof Kazhdan Haviv Enriquez Ševera

Genus.





example [BN2]

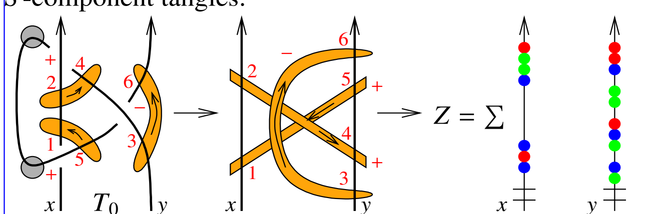
a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



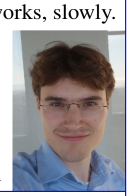
Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m_k^{ij}\colon A_i\otimes A_j \to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory ...works, slowly.

Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

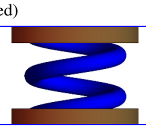
$$\left\{\begin{matrix}\text{ordered} & \text{perturbed} \\ \text{Gaussian formulas} & \end{matrix}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$


van der Veen

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org



This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at .../Indiana-1611/, .../LesDiablerets-1608/