

**1-Smidgen  $sl_2$**  Let  $\mathfrak{g}_1$  be the 4-dimensional Lie algebra  $\mathfrak{g}_1 = \langle h, e', l, f \rangle$  over the ring  $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ , with  $h$  central and with  $[f, l] = f$ ,  $[e', l] = -e'$ , and  $[e', f] = h - 2\epsilon l$ . Over  $\mathbb{Q}$ ,  $\mathfrak{g}_1$  is a **solvable approximation of  $sl_2$** :  $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon e', \epsilon l, \epsilon f \rangle \supset \langle h, \epsilon h, \epsilon e', \epsilon l, \epsilon f \rangle \supset 0$ . Pragmatics: declare  $\deg(h, e', l, f, \epsilon) = (1, 1, 0, 0, 1)$  and set  $t := e^h$  and  $e := (t - 1)e'/h$ .

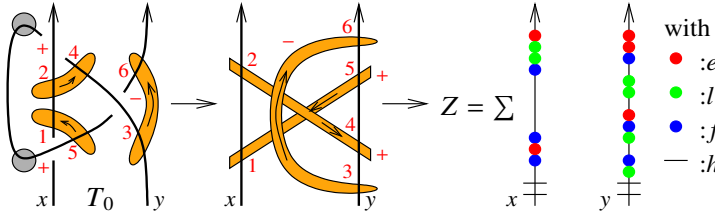
**How did it arise?**  $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$ , where  $\mathfrak{b}^+ = \langle l, f \rangle/[f, l] = f$  is a Lie bialgebra with  $\delta: \mathfrak{b}^+ \rightarrow \mathfrak{b}^+ \otimes \mathfrak{b}^+$  by  $\delta: (l, f) \mapsto (0, l \wedge f)$ . Going back,  $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f \rangle/\dots$ . **Idea.** Replace  $\delta \rightarrow \epsilon\delta$  over  $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$ . At  $k = 1$ , get  $[f, l] = f$ ,  $[f, h'] = -\epsilon f$ ,  $[l, e'] = e'$ ,  $[h', e'] = -\epsilon e'$ ,  $[h', l] = 0$ , and  $[e', f] = h' - \epsilon l$ . Now note that  $h' + \epsilon l$  is central, so switch to  $h := h' + \epsilon l$ . This is  $\mathfrak{g}_1$ .

**Ordering Symbols.**  $\odot$  (*poly* | *specs*) plants the variables of *poly* in  $\hat{S}(\oplus_{\mathfrak{g}})$  along  $\hat{U}(\mathfrak{g})$  according to *specs*. E.g.,

$$\odot(e_1 e^{\epsilon^3} l_1^3 l_2 f_3^9 | f_3 l_1 e_1 \epsilon_3 l_2) = f^9 l^3 e e^{\epsilon} l \in \hat{U}(\mathfrak{g}).$$

This enables the description of elements of  $\hat{U}(\mathfrak{g})$  using commutative polynomials / power series. In  $\mathfrak{g}_1$ , no need to specify  $h/t$ .

**Algebras and Invariants.** Given any unital algebra  $A$  (even better if  $A$  is Hopf; typically,  $A \sim \hat{U}(\mathfrak{g})$ ), appropriate **orange**  $R \in A \otimes A$ , and appropriate cuaps  $\in A$ , get an  $A^{\otimes S}$ -valued invariant of pure  $S$ -component tangles:



**What we didn't say** (more, including videos, in  $\omega\epsilon\beta$ /Talks).

- $\rho_1$  is “line” in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- $\rho_1$  extends to “rotational virtual tangles” and is a projection of the universal finite type invariant of such.
- $\rho_1$  seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- $\rho_1$  leads to many questions and a very long to-do list. Years of work, many papers ahead. Have fun!

**Demo Programs.**

```

CF [E_] := Module[{vars = Union@Cases[E, e_ | 1_ | f_, \infty]},
  If[vars === {}, Factor[E],
    Total[CoefficientRules[E, vars] /.
      (p_ -> c_) => Factor[c] Times @@ (vars^p) ] ] ];
CF [E_#] := CF /@ E;
E [i_, j_, s_] := E [1, (-1)^s 1_j, (-t)^s e_i f_j,
  t^s e_i 1_{(1+s) i-s j} f_j + (-1)^s 1_i 1_j + (-t^2)^s e_i^2 f_j^2/4];
E [i_, s_] := E [1, \theta, \theta, s 1_i];
E /: E [1, L1_, Q1_, P1_] E [1, L2_, Q2_, P2_] :=
  E [1, L1 + L2, Q1 + Q2, P1 + P2];
  
```

$\omega\epsilon\beta$ /Demo

**Formatting**  
(prints differ ☺)

**Preparation**

**z1 =** ( $\mathbb{E}[1, 11, \theta]$   $\mathbb{E}[4, 2, -1]$   $\mathbb{E}[15, 5, \theta]$  **Preparing the Trefoil**  
 $\mathbb{E}[6, 8, -1]$   $\mathbb{E}[9, 16, \theta]$   $\mathbb{E}[12, 14, -1]$   $\mathbb{E}[3, -1]$   $\mathbb{E}[7, +1]$   
 $\mathbb{E}[10, -1]$   $\mathbb{E}[13, +1]$ )

$$\mathbb{E} \left[ 1, -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16}, \right. \\ \left. - \frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16}, \right. \\ \left. - \frac{e_2^2 f_2^2}{4 t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4 t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4 t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \right. \\ \left. \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + \right. \\ \left. l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16} \right]$$

$DP_{x \rightarrow \partial \alpha, y \rightarrow \partial \beta} [P_] [f_] :=$  **Differential Polynomials**

**Total[CoefficientRules[P, {x, y}] /.** (Implementing  $P(\partial_\alpha, \partial_\beta)(f)$ )  
 $(\{m_, n_ \} \rightarrow c_) \Rightarrow c D[f, \{\alpha, m\}, \{\beta, n\}]$

$S_{1_j} (x:e|f)_i \rightarrow k_- [\mathbb{E}[\omega_-, L_-, Q_-, P_-]] :=$  **le and fl Sorts**

**With** [ $\{\lambda = \partial_{1_j} L, \alpha = \partial_{x_i} Q, q = e^x \beta x_k + \gamma 1_k\}$ , **CF** [ $\mathbb{E}[\omega, L / . 1_j \rightarrow 1_k, t^\lambda \alpha x_k + (Q / . x_i \rightarrow \theta),$   
 $e^{-q} DP_{1_j \rightarrow \partial \alpha, x_i \rightarrow \partial \beta} [P] [e^q] / . \{\beta \rightarrow \alpha / \omega, \gamma \rightarrow \lambda \text{Log}[t]\}$  ] ]];

$$\Delta [k_-] := ((t - 1) (2 (\alpha \beta + \delta \mu)^2 - \alpha^2 \beta^2) - 4 e_k 1_k f_k \delta^2 \mu^2 - \\ \delta (1 + \mu) (f_k^2 \alpha^2 + e_k^2 \beta^2) - e_k^2 f_k^2 \delta^3 (1 + 3 \mu) - \\ 2 (\alpha \beta + 2 \delta \mu + e_k f_k \delta^2 (1 + 2 \mu) + 2 1_k \delta \mu^2) (f_k \alpha + e_k \beta) - \\ 4 (1_k \mu^2 + e_k f_k \delta (1 + \mu)) (\alpha \beta + \delta \mu)) (1 + t) / 4;$$

**The Λόγος**

$S_{f_i e_j \rightarrow k_-} [\mathbb{E}[\omega_-, L_-, Q_-, P_-]] :=$  **fe Sorts**

**With** [ $\{q = ((1 - t) \alpha \beta + \beta e_k + \alpha f_k + \delta e_k f_k) / \mu\}$ , **CF** [ $\mathbb{E}[\mu \omega, L, \mu \omega q + \mu (Q / . f_i | e_j \rightarrow \theta),$   
 $\mu^4 e^{-q} DP_{f_i \rightarrow \partial \alpha, e_j \rightarrow \partial \beta} [P] [e^q] + \omega^k \Delta [k_-] / . \mu \rightarrow 1 + (t - 1) \delta / .$   
 $\{\alpha \rightarrow \omega^{-1} (\partial_{f_i} Q / . e_j \rightarrow \theta), \beta \rightarrow \omega^{-1} (\partial_{e_j} Q / . f_i \rightarrow \theta),$   
 $\delta \rightarrow \omega^{-1} \partial_{f_i, e_j} Q\}$  ] ]];

$m_{i, j \rightarrow k_-} [Z \_E] :=$  **Elf Merges** **Module** [ $\{x, z\}$ ,

**CF** [ $Z // S_{f_i e_j \rightarrow x} // S_{1_i e_x \rightarrow x} // S_{f_x 1_j \rightarrow x} / . z_{-i|j|x} \rightarrow z_k$  ] ]

**(Do** [ $z1 = z1 // m_{1, k+1}, \{k, 2, 16\}$  ]]; **z1**) **Rewriting the Trefoil**

(by merging 16 elves)

$$\mathbb{E} \left[ \frac{1-t+t^2}{t}, \theta, \theta, \frac{(-1+t) (1-t+t^2)^2 (1-t+2t^2)}{t^3} - \right. \\ \left. \frac{2 (1+t) (1-t+t^2)^3 e_1 f_1}{t^4} - \frac{2 (-1+t) (1+t) (1-t+t^2)^3 1_1}{t^4} \right]$$

**Readout**

$$\rho_1 [\mathbb{E}[\omega_-, \_, \_, P_-]] := CF \left[ \frac{t ((P / . e_- | 1_- | f_- \rightarrow \theta) - t \omega^3 (\partial_t \omega))}{(t - 1)^2 \omega^2} \right]$$

$\rho_1 [z1] //$  **Expand**

$\rho_1(3i)$

$$\frac{1}{t} + t$$

**References.**

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis,  $\omega\epsilon\beta$ /Ov.  
 [Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.  
 [Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.  
 [Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

diagram	$n_k^a$	Alexander's $\omega^+$	genus / ribbon	diagram	$n_k^a$	Alexander's $\omega^+$	genus / ribbon
		Today's / Rozansky's $\rho_1^+$	unknotting number / amphicheiral			Today's / Rozansky's $\rho_1^+$	unknotting number / amphicheiral
	$0_1^a$	1	0 / ✓		$3_1^a$	$t - 1$	1 / ✗
	$0$		0 / ✓		$t$		1 / ✗
	$4_1^a$	$3 - t$	1 / ✗		$5_1^a$	$t^2 - t + 1$	2 / ✗
	$0$		1 / ✓		$2t^3 + 3t$		2 / ✗