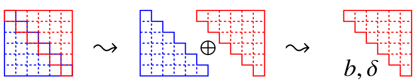
What else can you do with solvable approximations?

Abstract. Recently, Roland van der Veen and myself found that there are sequences of solvable Lie algebras "converging" to any given semi-simple Lie algebra (such as sl_2 or sl_3 or $E8$). Certain computations are much easier in solvable Lie algebras; in particular, using solvable approximations we can compute in polynomial time certain projections (originally discussed by Rozansky) of the knot invariants arising from the Chern-Simons-Witten topological quantum field theory. This provides us with the first strong knot invariants that are computable for truly large knots.
But sl_2 and sl_3 and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

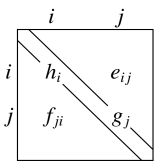
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\text{◹}, b, \delta)$:



$b(\text{◹}) = b\colon \text{◹} \otimes \text{◹} \to \text{◹}$

$b(\text{◺}) \rightsquigarrow \delta\colon \text{◹} \to \text{◹} \otimes \text{◹}$

Now define $gl_n^\epsilon := \mathcal{D}(\text{◹}, b, \epsilon\delta)$. Schematically, this is $[\text{◹},\text{◹}] = \text{◹}$, $[\text{◺},\text{◺}] = \epsilon\text{◺}$, and $[\text{◹},\text{◺}] = \text{◺} + \epsilon\text{◹}$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$

$$\begin{aligned}[e_{ij}, f_{kl}] = {} & \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il}) \\ & -\delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})\end{aligned}$$

$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$

$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

Solvable Approximation. At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, ..., $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

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In[1]:= MatrixExp[( a b
                    c d )] // FullSimplify // MatrixForm     Enter
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Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

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In[2]:= MatrixExp[( a b
                    0 c )] // MatrixForm
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Out[2]//MatrixForm=

$$\begin{pmatrix} e^a & \frac{b\,(e^a - e^c)}{a-c} \\ 0 & e^c \end{pmatrix}$$

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In[3]:= MatrixExp[( a1 b1
                    0  c1 )].MatrixExp[( a2 b2
                                         0  c2 )] //
          MatrixLog // PowerExpand // Simplify //
        MatrixForm                                     Enter
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Question. What else can you do with solvable approximation? Chern-Simons-Witten theory is often "solved" using ideas from conformal field theory and using quantization of various moduli spaces. Does it make sense to use solvable approximation there too? Elsewhere in physics? Elsewhere in mathematics?
See Also. Talks at George Washington University [ωεβ/gwu], Indiana [ωεβ/ind], and Les Diablerets [ωεβ/ld], and a University of Toronto "Algebraic Knot Theory" class [ωεβ/akt].

Chern-Simons-Witten. Given a knot $\gamma(t)$ in $\mathbb{R}^3$ and a metrized Lie algebra $\mathfrak{g}$, set $Z(\gamma) :=$

$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, e^{ik\,cs(A)} PExp_\gamma(A),$$

where $cs(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$ and

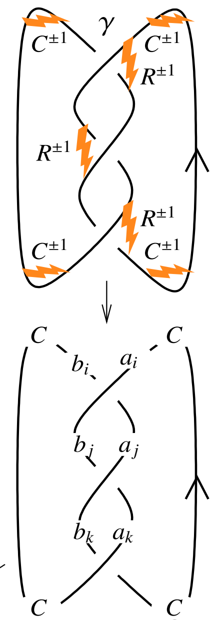
$$PExp_\gamma(A) := \prod_0^1 \exp(\gamma^* A) \in \mathcal{U} = \hat{\mathcal{U}}(\mathfrak{g}),$$

and $\mathcal{U}(\mathfrak{g}) := \langle\text{words in } \mathfrak{g}\rangle/(xy - yx = [x,y])$. In a favourable gauge, one may hope that this computation will localize near the crossings and the bends, and all will depend on just two quantities,

$$R = \sum a_i \otimes b_i \in \mathcal{U}\otimes\mathcal{U} \quad \text{and} \quad C \in \mathcal{U}.$$

This was never done formally, yet R and C can be "guessed" and all "quantum knot invariants" arise in this way. So for the trefoil,

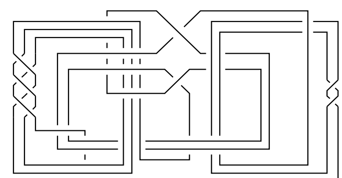
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$



But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?
Solution 1, Representation Theory. Choose a finite dimensional representation ρ of $\mathfrak{g}$ in some vector space V. By luck and the wisdom of Drinfel'd and Jimbo, $\rho(R) \in V^*\otimes V^*\otimes V\otimes V$ and $\rho(C) \in V^*\otimes V$ are computable, so Z is computable too. But in exponential time!

Ribbon=Slice?

Gompf, Scharlemann, Thompson



Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, where $\mathfrak{g}_k = sl_2^k$ (or a similar algebra); everything is expressible using low-degree polynomials in a small number of variables, hence everything is poly-time computable!

Example 0. Take $\mathfrak{g}_0 = sl_2^0 = \mathbb{Q}\langle h,e,l,f\rangle$, with h central and $[f,l] = f$, $[e,l] = -e$, $[e,f] = h$. In it, using normal orderings,

$$R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h}ef\right) \mid e\otimes lf\right), \quad \text{and,}$$

$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu\delta ef} \mid ef\right) \quad \text{with } \nu = (1 + h\delta)^{-1}.$$

Example 1. Take $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ and $\mathfrak{g}_1 = sl_2^1 = R\langle h,e,l,f\rangle$, with h central and $[f,l] = f$, $[e,l] = -e$, $[e,f] = h - 2\epsilon l$. In it,

$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)e^{\nu\delta ef} \mid elf\right), \quad \text{where } \Lambda \text{ is}$$

$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l.$
Fact. Setting $h_i = h$ (for all i) and $t = e^h$, the $\mathfrak{g}_1$ invariant of any tangle T can be written in the form

$$Z_{\mathfrak{g}_1}(T) = \mathbb{O}\left(\omega^{-1}e^{hL+\omega^{-1}Q}(1 + \epsilon\omega^{-4}P) \mid \bigotimes_i e_i l_i f_i\right),$$

where L is linear, Q quadratic, and P quartic in the $\{e_i, l_i, f_i\}$ with ω and all coefficients polynomials in t. Furthermore, everything is poly-time computable.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/McGill-1702/