

The $s/2$ Example. Let $g^\epsilon = \langle h, e, l, f \rangle / ([h, \cdot] = 0, [e, l] = -e, [f, l] = f, [e, f] = h - 2\epsilon l)$ and let $g_k = g^\epsilon / (\epsilon^{k+1} = 0)$.

The Main g_k Theorem. The g_k -invariant of any S -component tangle T can be written in the form

$$Z(T) = \bigcirc \left(\omega e^{L+Q+P} : \bigotimes_{i \in S} e_i l_i f_i \right),$$

where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i := e^{h_i}$), where $L = \sum a_{ij} h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients, where $Q = \sum b_{ij} e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij} , and where P is a polynomial in $\{\epsilon, e_i, l_i, f_i\}$ (with scalar coefficients) whose ϵ^d -term is of degree at most $2d + 2$ in $\{e_i, \sqrt{l_i}, f_i\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i , the invariant $Z(T)$ is poly-time computable.

The Main g_k Lemma. The following “re-ordering relations” hold:

$$\bigcirc (e^{\gamma l + \beta e} : le) = \bigcirc (e^{\gamma l + \beta e} : el) \quad (\text{and similarly for } fl \rightarrow lf),$$

$\bigcirc (e^{\beta e + \alpha f + \delta \epsilon f} : fe) = \bigcirc (v e^{\nu(-\alpha \beta h + \beta e + \alpha f + \delta \epsilon f) + \lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)} : elf)$, with $v = (1 + h\delta)^{-1}$ and where $\lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is some fixed polynomial of degree at most $2k + 2$ in $\epsilon, e, \sqrt{l}, f, \alpha, \beta, \delta$, with scalar coefficients.

Demo Programs.

CF [\mathcal{E}_-] := **Module** [$\{\text{vars} = \text{Union@Cases}[\mathcal{E}, e_ | l_ | f_ , \infty]\}$,

If [$\text{vars} == \{\}$, **Factor** [\mathcal{E}],

Total [**CoefficientRules** [\mathcal{E} , vars] /.

$(p_ \rightarrow c_) \Rightarrow \text{Factor}[c] \text{ Times} @@ (\text{vars}^p)]]]$;

CF [\mathcal{E}_E] := **CF** /@ \mathcal{E} ;

E [$i_ , j_ , s_$] := **E** [$1, (-1)^s l_j, (-t)^s e_i f_j,$

$t^s e_i l_{(1+s) i-s j} f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2 / 4$];

E [$i_ , s_$] := **E** [$1, \theta, \theta, s l_i$];

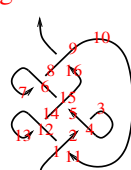
E /: **E** [$1, L1_ , Q1_ , P1_$] **E** [$1, L2_ , Q2_ , P2_$] :=

E [$1, L1 + L2, Q1 + Q2, P1 + P2$];

z1 = (**E** [$1, 11, \theta$] **E** [$4, 2, -1$] **E** [$15, 5, \theta$] \times **Preparing the Trefoil**

E [$6, 8, -1$] **E** [$9, 16, \theta$] **E** [$12, 14, -1$] \times

E [$3, -1$] **E** [$7, +1$] **E** [$10, -1$] **E** [$13, +1$])



$$\begin{aligned} & \mathbf{E} \left[1, -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16}, \right. \\ & - \frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16}, \\ & - \frac{e_4^2 f_2^2}{4 t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4 t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4 t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \\ & \left. \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + \right. \\ & \left. l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16} \right] \end{aligned}$$

DP $x_ \rightarrow \partial_{\alpha}, y_ \rightarrow \partial_{\beta}$ [P_-] [f_-] := **Differential Polynomials**

Total [**CoefficientRules** [$P, \{x, y\}$] /. (Implementing $P(\partial_{\alpha}, \partial_{\beta})(f)$)

$(\{m_ , n_ \} \rightarrow c_) \Rightarrow c \mathbf{D}[f, \{\alpha, m\}, \{\beta, n\}]]]$

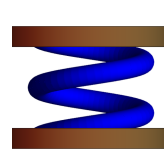
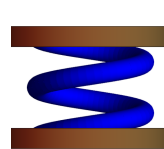


diagram	n_k^t Alexander's ω^+	genus / ribbon	diagram	n_k^t Alexander's ω^+	genus / ribbon
	Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral		Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral
	0_1^a 1	0 / ✓		3_1^a $t - 1$	1 / ✗
	0	0 / ✓		t	1 / ✗
	4_1^a $3 - t$	1 / ✗		5_1^a $t^2 - t + 1$	2 / ✗
	0	1 / ✓		$2t^3 + 3t$	2 / ✗

Video and more at <http://www.math.toronto.edu/~drorbn/Talks/Toulouse-1705/>