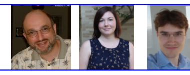
The Dogma is Wrong

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at ωεβ/talks.



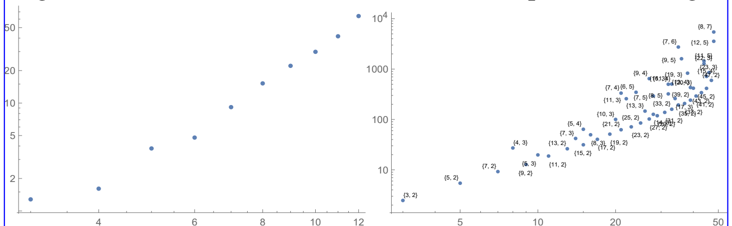
Thanks for the invitation!

ωεβ:=http://drorbn.net/Toulouse-1705/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

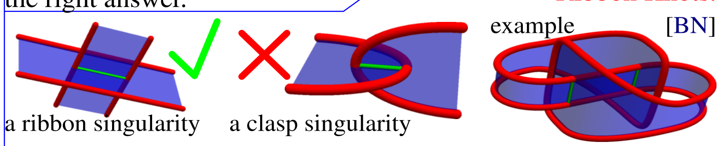
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:

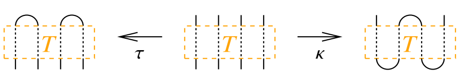


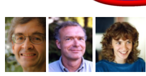
Power. On the 250 knots with at most 10 crossings, the pair (ω,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg\rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.





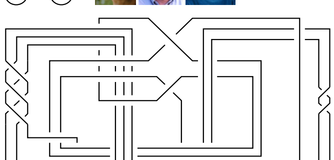


Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n & \\ \mathcal{T}_{2n} & \xrightarrow{\ z\ } & \mathcal{A}_{2n} & & \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 & \end{array}$$

[Vo]: Works for Alexander!

with $\mathcal{R} := \kappa(\tau^{-1}(1))$



$$A^+ = -t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$$

$$\rho_1^+ = 5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$$

Faster is better, leaner is meaner!

dog·ma (dôg′mə, dŏg′-) — The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

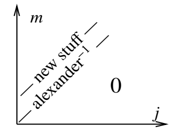
2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

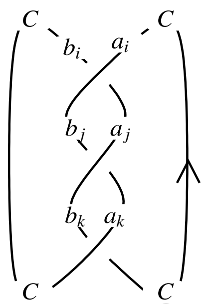
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar)=1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra A (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i\otimes b_i \in A\otimes A \quad\text{and}\quad C\in A,$$

form

$$Z = \sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

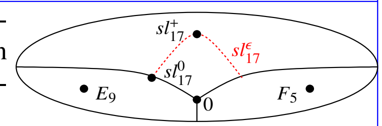
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m_k^{ij}\circlearrowright\{\mathcal{F}_S\}\xrightarrow{\ \mathbb{E}\ }\{A^{\otimes S}\}\circlearrowleft m_k^{ij}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.



Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
In[1]:= MatrixExp[( a b
                    c d )] // FullSimplify // MatrixForm     Enter
```

Yet in solvable algebras, exponentiation is fine and even BCH, $z=\log(e^xe^y)$, is bearable:

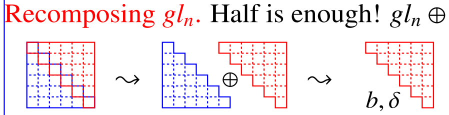
```
In[2]:= MatrixExp[( a b
                    0 c )] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} e^a & \frac{b(e^a-e^c)}{a-c} \\ 0 & e^c \end{pmatrix}$$

```
In[3]:= MatrixExp[( a1 b1
                    0  c1 )].MatrixExp[( a2 b2
                                         0  c2 )] //
          MatrixLog // PowerExpand // Simplify //
        MatrixForm                                    Enter
```

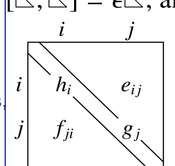
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b\colon ◹\otimes◹\to◹$

$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is [◹, ◹] = ◹, [◺, ◺] = ϵ◺, and [◹, ◺] = ◺ + ϵ◹. In detail, it is



$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj} \qquad [f_{ij},f_{kl}]=\epsilon\delta_{jk}f_{il}-\epsilon\delta_{li}f_{kj}$$

$$[e_{ij},f_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}e_{il}+\delta_{il}(h_i+\epsilon g_i)/2+\delta_{i>l}f_{il}) -\delta_{li}(\epsilon\delta_{k<j}e_{kj}+\delta_{kj}(h_j+\epsilon g_j)/2+\delta_{k>j}f_{kj})$$

$$[g_i,e_{jk}]=(\delta_{ij}-\delta_{ik})e_{jk} \qquad [h_i,e_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})e_{jk}$$

$$[g_i,f_{jk}]=(\delta_{ij}-\delta_{ik})f_{jk} \qquad [h_i,f_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})f_{jk}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Toulouse-1705/